Strongly not relatives Kähler manifolds

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According to [3], two Kähler manifolds are called *relatives* when they share a common Kähler submanifold, i.e. if a complex submanifold of one of them with the induced metric is biholomorphically isometric to a complex submanifold of the other one with the induced metric. In his seminal paper [1], Calabi determined a criterion which characterizes Kähler manifolds admitting a Kähler immersion into finite or infinite dimensional complex space forms. The main tool he introduced is the *diastasis function* associated to a real analytic Kähler manifold, namely a particular Kähler potential characterized by being invariant under pull–back through a holomorphic map. Thanks to this property, the diastasis plays a key role in studying when two Kähler manifolds are relatives.

In [4] I have been interested in characterizing Kähler manifolds that are *strongly not relative* to any projective Kähler manifold, i.e. those Kähler manifolds that do not share a Kähler submanifold with any projective Kähler manifold even when their metric is rescaled by the multiplication by a positive constant. Here with projective Kähler manifold I mean a Kähler manifold admitting a local holomorphic and isometric (from now on *Kähler*) immersion into the finite dimensional complex projective space $\mathbb{C}\mathrm{P}^N$ endowed with its Fubini–Study metric g_{FS}. The sufficient conditions I stated are given in terms of the existence of a *full* Kähler immersion into the infinite dimensional complex projective space $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$, where *full* means that the image of the Kähler manifold is not contained into a lower dimensional $(\mathbb{C}\mathrm{P}^N, g_{FS})$ for any $N < \infty$.

The first result achieved in [4] can be stated as follows:

Theorem 1. *Let (M, g) be a Kähler manifold such that $(M, \beta g)$ admits a full Kähler immersion into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$ for any $\beta > \beta_0 \geq 0$. If (M, g) and $(\mathbb{C}\mathrm{P}^n, g_{FS})$ are not relatives for any $n < \infty$, then (M, g) is strongly not relative to any projective Kähler manifold.*

Observe that in general there are not reasons for a Kähler manifold which is not relative to another Kähler manifold to remain so when its metric is rescaled. For example, consider that the complex projective space $(\mathbb{C}\mathrm{P}^2, c\, g_{FS})$ where g_{FS} is the Fubini–Study metric, for $c = \frac{2}{3}$ is not relative to $(\mathbb{C}\mathrm{P}^2, g_{FS})$, while for positive integer values of c it is (see [2] for a proof).

In order to state the second result achieved in [4], consider a d-dimensional Kähler manifold (M, g) which admits global coordinates $\{z_1, \ldots, z_d\}$ and denote by M_j the 1-dimensional submanifold of M defined by:

$$M_j = \{z \in M |\, z_k = 0\ \forall\, k \neq j\}.$$

When exists, a Kähler immersion $f \colon M \to \mathbb{C}\mathrm{P}^\infty$, $f^* g_{FS} = g$, is said to be *transversally full* when for any $j = 1, \ldots, d$, the immersion restricted to M_j is full into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$.

Theorem 2. *Let (M, g) be a Kähler manifold which admits a full Kähler immersion into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$ through a transversally full map. If for any $\alpha \geq \alpha_0 > 0$, $(M, \alpha\, g)$ admits a full Kähler immersion into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$, then (M, g) is strongly not relative to any projective Kähler manifold.*

The proof of both theorems are completely based on the properties of the diastasis function associated to a Kähler manifold.

Finally, Theorem 1 and Theorem 2 are applied to two 1-parameter families of Hartogs-type domains, proving that they are strongly not relative to any projective Kähler manifold.

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