The diastasis function of the Cigar metric

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In his seminal paper [1], E. Calabi provides a criterion for a Kähler manifold (M, g) to admit a holomorphic and isometric (from now on Kähler) immersion into a complex space form, finite or infinite dimensional. Recall that a complex space form, that we assume to be complete and simply connected, up to homotheties can be of three types, according to the sign of the constant holomorphic sectional curvature:

(i) the complex Euclidean space $\mathbb{C}^N$ of complex dimension $N \leq \infty$, endowed with the flat metric. Here $\mathbb{C}^\infty$ denotes the Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences w_j, $j = 1, 2, \ldots$, $w_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |w_j|^2 < +\infty$.

(ii) the complex projective space $\mathbb{C}\mathrm{P}^N$ of complex dimension $N \leq \infty$, with the Fubini-Study metric g_{FS}.

(iii) the complex hyperbolic space $\mathbb{C}\mathrm{H}^N$ of complex dimension $N \leq \infty$, that is the unit ball $B \subset \mathbb{C}^N$ given by $B = \left\{(z_1, \ldots, z_N) \in \mathbb{C}^N,\ \sum_{j=1}^N |z_j|^2 < 1\right\}$ endowed with the hyperbolic metric.

The criterion Calabi states is given in terms of the *diastasis function* associated to the metric g, defined as follows. By duplicating the variables z and $\bar{z}$, a Kähler potential Φ around $p \in M$ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times M$. The diastasis function is defined by the formula:

$$D^g(p, q) = \tilde{\Phi}(p, \bar{p}) + \tilde{\Phi}(q, \bar{q}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}),$$

for $(p, q) \in U$. Once p is fixed and z are coordinates around it, $D^g(p, q) = D^g_p(z)$ is a Kähler potential in a neighborhood of p. The existence of a local Kähler immersion into a complex space form depends only on the derivatives with respect to z and $\bar{z}$ of $D^g_p(z)$, evaluated at p. In particular, when (M, g) is a complex curve endowed with a radial metric g and the ambient space is the complex projective space, the criterion reads as follows:

Calabi's criterion. *Let (M, g) be a real analytic Kähler curve such that in a coordinate system $\{z\}$ centered at $p \in M$, $D^g_p(z)$ depends only on $|z|^2$. Then (M, g) admits a local Kähler immersion into $\mathbb{C}\mathrm{P}^{N \leq \infty}$ iff for any $n > 0$:*

$$\frac{\partial^{2n} \exp\left(D_0(|z|^2)\right)}{\partial z^n \partial \bar{z}^n}|_0 \geq 0.$$

Further, the image of M is not contained in any totally geodesic submanifold of the N-dimensional ambient space iff the number of derivatives in the above inequality different from zero are exactly N.

A local Kähler immersion can be extended to a global one iff for each point $p \in M$, the maximal analytical extension of $D^g_p(z)$ is single valued, condition which is fulfilled when M is simply connected.

In [3], we have been interested in studying how the multiplication of the Kähler metric g by a positive constant c affects the existence of a (local) Kähler immersion of (M, cg) into complex space forms. More precisely, we proved the following theorem:

Theorem 1. *Let $g = \frac{1}{1+|z|^2} dz \otimes d\bar{z}$ be the Cigar metric on $\mathbb{C}$. Then the diastasis function of the metric g is globally defined and positive on $\mathbb{C} \times \mathbb{C}$ and $(\mathbb{C}, cg)$ cannot be (locally) Kähler immersed into any complex space form for any $c > 0$.*

Observe that when one studies Kähler immersions into the complex Euclidean space, the multiplication of the metric g by c is harmless. In fact, if $f : M \to \mathbb{C}^N$, $N \leq \infty$, satisfies $f^*(g_0) = g$ then $(\sqrt{c} f)^*(g_0) = cg$. On the other hand, when the ambient space is the complex projective space the situation is completely different. A very interesting example in this sense is given by Cartan domains Ω, i.e. irreducible bounded homogeneous domains of $\mathbb{C}^d$ endowed with their Bergman metric g_B. The existence of a Kähler immersion of $(\Omega, c\, g_B)$ into $\mathbb{C}\mathrm{P}^\infty$ depends firmly on c and it is strictly related to the Wallach set $W(\Omega)$ of Ω (see [2]). More precisely, if γ is the genus of Ω, then (Ω, cg_B), $c > 0$ admits a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ if and only if $c\gamma$ belongs to $W(\Omega) \setminus \{0\}$.

(We refer the reader to [2] and refernces therein for more detalis about $W(\Omega)$)

In view of the case of Hermitian symmetric spaces it is natural and interesting to exhibit examples of manifolds (M, cg) that do not admit a Kähler immersion into any complex space form for any value of $c > 0$. At the end of his paper Calabi himself provides the following two examples (in the first one M is compact and in the second one M is noncompact and complete).

Example 1. Consider the product $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ endowed with the metric $g = b_1 g_{FS} \oplus b_2 g_{FS}$, with b_1, b_2 positive real numbers such that b_2/b_1 is irrational. Then $(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, cg)$ does not admit a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ for any value of c. In fact, in [1, Th.13], Calabi proves that $(\mathbb{C}\mathrm{P}^n, cg_{FS})$ admits a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ iff $1/c$ is a positive integer, and this property cannot be fulfilled by both $1/cb_1$ and $1/cb_2$.

Example 2. Consider on $\mathbb{C}$ the metric g whose associate Kähler form ω is given by: $\omega = (4\cos(z - \bar{z}) - 1)\, dz \wedge d\bar{z}$. A Kähler immersion of $(\mathbb{C}, cg)$ into $\mathbb{C}\mathrm{P}^\infty$ is not possible since the diastasis:

$$D(p,q) = 4\left[\cos(p - \bar{p}) + \cos(q - \bar{q}) + \right. \\ \left. - \cos(p - \bar{q}) - \cos(q - \bar{p})\right] - |p - q|^2,$$

takes negative values, e.g. for $q = p + 2\pi$.

Observe that in the first example, the Kähler form ω associated to g is not integral, while in the second one the diastasis associated to g is negative at some points. The importance of the Cigar metric studied in Theorem 1 relies on the fact that, unlike the previous two examples, it does not present geometrical obstructions to the existence of a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ that put aside the role of c. More precisely, it is an example of real analytic Kähler manifold (M, cg) which cannot be Kähler immersed into any (finite or infinite dimensional) complex space form for any $c > 0$ and satisfy:

(i) the Kähler form ω associated to g is integral;

(ii) the diastasis associated to g is globally defined on $M \times M$ and positive.

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