Equations of Nonlinear Schrödinger type

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1. Inverse Scattering Transform for NLS systems with non-zero boundary conditions

In [1] we developed the inverse scattering transform (IST) for the defocusing nonlinear Schrodinger (NLS) equation:

$$iq_t = q_{xx} - 2|q|^2 q$$

with fully asymmetric non-zero boundary conditions as $x \to \pm\infty$, (i.e., when the limiting values of the solution at space infinities have different non zero moduli). The theory is formulated without making use of Riemann surfaces, and instead by dealing explicitly with the branched nature of the eigenvalues of the associated scattering problem. For the direct problem, we gave explicit single-valued definitions of the Jost eigenfunctions and scattering coefficients over the whole complex plane, and we characterized their discontinuous behavior across the branch cut arising from the square root behavior of the corresponding eigenvalues. We posed the inverse problem as a Riemann-Hilbert problem on an open contour, and we reduced the problem to a standard set of linear integral equations. Finally, for comparison purposes, we also presented the single-sheet, branch cut formulation of the inverse scattering transform for the initial value problem with symmetric (equimodular) non-zero boundary conditions, as well as for the initial value problem with one-sided non-zero boundary conditions, and we also briefly described the formulation of the inverse scattering transform when a different choice is made for the location of the branch cuts.

In [2] we presented a rigorous theory of the IST for the three-component defocusing NLS equation with initial conditions approaching constant values with the same amplitude as $x \to \pm\infty$. The theory combines and extends to a problem with non-zero boundary conditions three fundamental ideas: (i) the tensor approach used by Beals, Deift and Tomei for the n-th order scattering problem, (ii) the triangular decompositions of the scattering matrix used by Novikov, Manakov, Pitaevski and Zakharov for the N-wave interaction equations, and (iii) a generalization of the cross product via the Hodge star duality, which, to the best of our knowledge, is used in the context of the IST for the first time in this work. The combination of the first two ideas allows us to rigorously obtain a fundamental set of analytic eigenfunctions. The third idea allows us to establish the symmetries of the eigenfunctions and scattering data. The results are used to characterize the discrete spectrum and to obtain exact soliton solutions, which describe generalizations of the so-called dark-bright solitons of the two-component NLS equation.

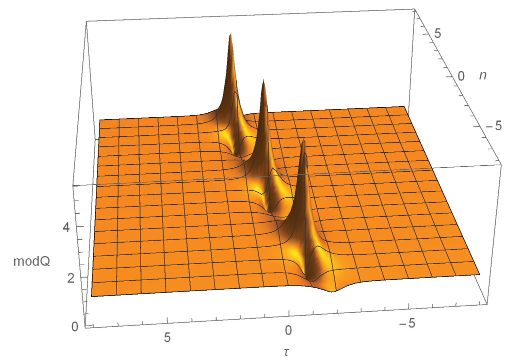

Figure 1. Discrete analog of the Tajiri-Watanabe soliton.

2. Inverse Scattering Transform and soliton solutions for the focusing Ablowitz-Ladik equation with non-zero boundary conditions

The focus of [3,4] is a semi-discrete (discrete in space, continuous in time) version of the NLS equation above. In general, a discretization of an integrable PDE is likely to be non-integrable. That is, even though the integrable PDE is the compatibility condition of a linear operator pair, one is not guaranteed to have a pair of linear equations corresponding to a generic discretization of the PDE. On the other hand, for the

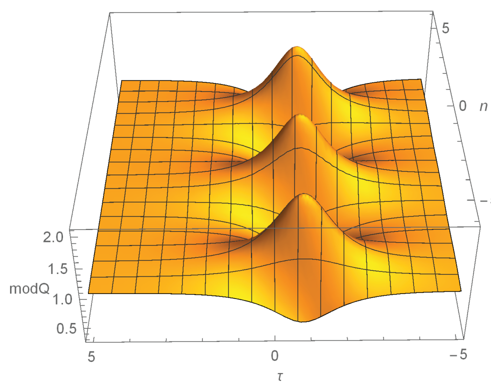

Figure 2. Discrete Akhmediev breather, periodic in n and homoclinic in τ.

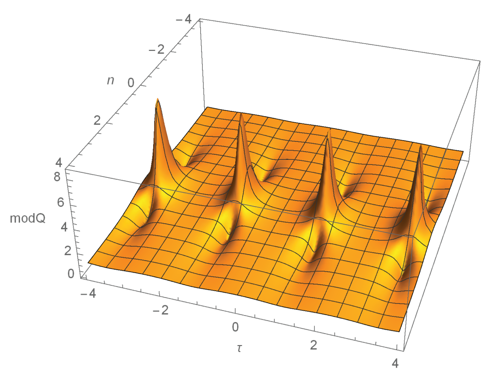

Figure 3. Discrete Kuznetsov-Ma solution, periodic in τ and homoclinic in n.

differential-difference equation

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) \mp |q_n|^2(q_{n+1} + q_{n-1})\,, \tag{1}$$

which is known in the literature as the Ablowitz-Ladik (AL) equation, and which is a $O(h^2)$ finite-difference approximation of NLS, there is such an associated operator pair.

Besides being used as a basis for numerical schemes for its continuous counterpart, the AL equation has also numerous physical applications, related to the dynamics of anharmonic lattices, self-trapping on a dimer, Heisenberg spin chains, etc.

In [3] we developed the IST for the focusing Ablowitz-Ladik [Eq.(1) with the + sign in front of

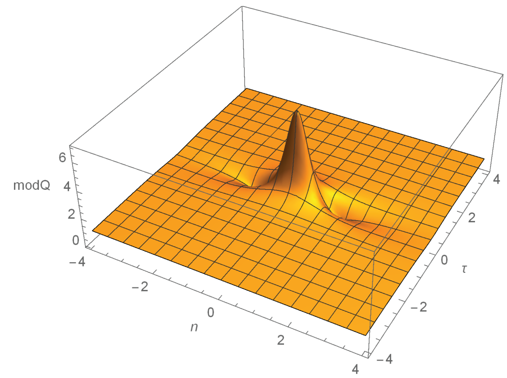

Figure 4. Discrete Peregrine solution.

the nonlinear term] equation with nonzero boundary conditions as $n \to \pm\infty$. Both the direct and the inverse problems were formulated in terms of a suitable uniform variable; the inverse problem is posed as a Riemann-Hilbert problem on a doubly-connected curve in the complex plane, and solved by properly accounting for the asymptotic dependence of eigenfunctions and scattering data on the Ablowitz-Ladik potential.

In [4] explicit soliton solutions are derived which are the discrete analog of the Tajiri-Watanabe and Kutznetsov-Ma solutions to the focusing NLS equation. Then, by performing suitable limits of the above solutions, discrete analog of the celebrated Akhmediev and Peregrine solutions were also obtained. These solutions, which have been derived by means of the Hirota bilinear method [6–8], are obtained here for the first time within the framework of the IST, thus also providing a spectral characterization of the solutions and a description of the singular limit process. Discrete breathers have a variety of applications. In particular, discrete rogue waves can be used as spatial energy concentrators in arrays of nonlinear waveguides [9]. As such, the results of this work may find practical applications in this context, as well as in all other physical settings where the AL solutions provide a good approximation for their continuous counterpart.

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