On solitonic surfaces

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In a joint work [1] with A. M. Grundland (CRM, Université de Montréal, Montréal and Dept. Math. Université du Québec, Trois-Rivières) and D. Levi (Dip. Mat. Fisica dell'Università Roma Tre, Sez. INFN di Roma Tre) we enlightened the relationships among three types of symmetries (gauge symmetries of the linear spectral problem, conformal transformations of the spectral parameter and generalized symmetries) of the immersion formula of 2D-surfaces associated integrable system. We proved that the mentioned symmetries can be expressed uniquely in terms of gauge transformations. We apply the theoretical results to surfaces associated to the nonlinear the sigma model.

The motivations of the work arises from the long standing relationship among Integrable Systems and Surfaces Geometry (see e.g. [2–16]). We say that a surface is integrable if its Gauss-Mainardi-Codazzi equations are integrable, i.e. if they can be represented as the compatibility conditions for some (non-fake) Linear Spectral Problem (LSP). This property yielded many new results concerning the intrinsic geometric properties of such surfaces and,their immersion functions.

The initial construction of surfaces related to completely integrable models makes use of the conformal invariance of the zero-curvature representation of the system with respect to the spectral parameter. Another approach is using gauge symmetries of the LSP. Moreover, by using the symmetries of the LSP associated to an integrable system, Fokas and Gel'fand [9,10] constructed families of soliton surfaces. Most recently, in a series of papers [11–14], a reformulation and extension of the Fokas-Gel'fand immersion formula has been performed through the formalism of generalized vector fields on jet spaces. This extension has provided the necessary and sufficient conditions for the existence of soliton surfaces in terms of the symmetries of the LSP of an integrable model.

Let us consider an integrable system in two independent variables x_1, x_2 and m dependent variables $u^k(x_1, x_2)$ written as integrability of the LSP ($\alpha = 1, 2$)

$$\begin{aligned} &D_2 U_1 - D_1 U_2 + [U_1, U_2] = 0, \\ &\partial_\alpha \Phi(x_1, x_2, \lambda) - U_\alpha([u], \lambda)\Phi(x_1, x_2, \lambda) = 0. \end{aligned} \tag{1}$$

Simultaneous infinitesimal deformation of (1) are

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{\Phi} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ \Phi \end{pmatrix} + \epsilon \begin{pmatrix} A_1 \\ A_2 \\ \Psi \end{pmatrix} + O(\epsilon^2). \tag{2}$$

They satisfy, at first order in ϵ, the equations

$$\begin{aligned} &D_\alpha \Psi = U_\alpha \Psi + A_\alpha \Phi, \\ &D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0. \end{aligned} \tag{3}$$

Theorem 1. *If the matrix functions $U_\alpha \in \mathfrak{g}$, $\Phi \in G$ of the LSP (1) and $A_\alpha \in \mathfrak{g}$ are linearly independent satisfying (3), then there exists (up to affine transformations) a 2D-surface with a $\mathfrak{g}$-valued immersion function $F([u], \lambda)$ such that the tangent vectors to this surface are given by*

$$D_\alpha F([u], \lambda) = \Phi^{-1} A_\alpha([u], \lambda)\Phi, \tag{4}$$

Theorem 2. *The linearly independent $\mathfrak{g}$-valued matrix functions*

$$\begin{aligned} A_\alpha([u], \lambda) = {} & \beta(\lambda) D_\lambda U_\alpha + (D_\alpha S + [S, U_\alpha]) \\ & + pr\omega_R U_\alpha + (pr\omega_R(D_\alpha \Phi - U_\alpha \Phi))\, \Phi^{-1}, \end{aligned} \tag{5}$$

where $\beta(\lambda)$ is an arbitrary scalar function of λ, $S = S([u], \lambda)$ is an arbitrary $\mathfrak{g}$-valued matrix function, $\omega_R = R^k[u]\partial_{u^k}$ is the evolutionary form of the generalized symmetries of (1), then there exists a 2D-surface with immersion function $F : \mathcal{N} \to \mathfrak{g}$ given by

$$F([u], \lambda) = \Phi^{-1} \left(\beta(\lambda) D_\lambda \Phi + S\Phi + pr\omega_R \Phi\right). \tag{6}$$

Special cases are : the ST immersion formula (when $S = 0, \omega_R = 0$) $F^{ST}([u], \lambda) = \beta(\lambda)\Phi^{-1}(D_\lambda \Phi) \in \mathfrak{g}$, the CD immersion formula (when $\beta = \omega_R = 0$) $F^{CD}([u], \lambda) = \Phi^{-1} S([u], \lambda)\Phi \in \mathfrak{g}$, and the FG immersion formula (when $\beta = 0, S = 0$) $F^{FG}([u], \lambda) =$

$\Phi^{-1}(\mathrm{pr}\omega_R\Phi) \in \mathfrak{g}$. In any case the (2.14) provides the tangent vectors and the unit normal vector to a 2D-surface

$$D_\alpha F = \Phi^{-1} A_\alpha \Phi \in \mathfrak{g}, \quad N = \frac{\Phi^{-1}[A_1, A_2]\Phi}{(\frac{1}{2}\mathrm{tr}[A_1, A_2]^2)^{1/2}} \in \mathfrak{g},$$

from which First and Second Fundamental forms are computed and the expressions for the Mean and Gaussian Curvatures are expressible in terms of U_α and A_α only.

Theorem 3. λ-CONFORMAL SYMMETRIES AND GAUGE TRANSFORMATIONS *A symmetry of (1) is a λ-conformal symmetry if and only if there exists a $\mathfrak{g}$-valued matrix function $S_1 = S_1([u], \lambda)$ which is a solution of the system of differential equations*

$$D_\alpha S_1 + [S_1, U_\alpha] = \beta(\lambda) D_\lambda U_\alpha. \tag{7}$$

Then, if the gauge function $S_1([u], \lambda)$ is known, by solving (7) we can determine the wavefunction Φ and consequently obtain the ST immersion formula for 2D-soliton surfaces. Therefore, the ST formula for immersion is equivalent to the CD immersion formula for the gauge S_1, satisfying the differential equation (7).

Theorem 4. GENERALIZED SYMMETRIES AND GAUGE TRANSFORMATIONS *A vector field ω_R is a generalized symmetry of (1) if and only if there exists a $\mathfrak{g}$-valued matrix function (gauge) $S_2 = S_2([u], \lambda)$ which is a solution of the system of differential equations.*

$$D_\alpha S_2 + [S_2, U_\alpha] = pr\omega_R U_\alpha + (pr\omega_R(D_\alpha \Phi - U_\alpha \Phi))\,\Phi^{-1}. \tag{8}$$

Comparing the above FG formula for immersion with the CD immersion formula, we find that the matrix function $S_2 = (\mathrm{pr}\omega_R\Phi)\Phi^{-1}$ satisfies (8) and, then, the FG formula is equivalent to the CD immersion formula.

Theorem 5. THE SYM-TAFEL'S VERSUS THE FOKAS-GEL'FAND IMMERSION FORMULA *Suppose that a generalized vector field written in the evolutionary form $\omega_R = R_k[u]\partial_{u^k}$ is a symmetry of the integrable PDE in (1) and that the $\mathfrak{g}$-valued matrices S_1 and S_2 satisfy the differential conditions (7) and (8), respectively. If the gauge S_2 is non-singular, then there exists a matrix $M = S_1 S_2^{-1}$ such that*

$$\beta(\lambda)(D_\lambda \Phi) = M(pr\omega_R \Phi). \tag{9}$$

Alternatively, if the gauge S_1 is non-singular, then there exists a matrix M^{-1} such that

$$(pr\omega_R \Phi) = M^{-1}\beta(\lambda)(D_\lambda \Phi). \tag{10}$$

Thus, also in this case we can find a mapping between the ST and the FG formulae. We summarise our result in the commutative diagram

$$\begin{tikzcd}
 & F^{ST} = \beta(\lambda)\Phi^{-1}(D_\lambda\Phi) \in \mathfrak{g} \arrow[d, shift left, "S_2 \circ S_1^{-1}"] \\
\Phi \in G \arrow[ur, "S_1 \in \mathfrak{g}"] \arrow[dr, "S_2 \in \mathfrak{g}"'] & \\
 & F^{FG} = \Phi^{-1}(\mathrm{pr}\omega_R\Phi) \in \mathfrak{g} \arrow[u, shift left, "S_1 \circ S_2^{-1}"]
\end{tikzcd}$$

Figure 1. Representation of the relations between the wavefunction $\Phi \in G$ and the $\mathfrak{g}$-valued ST and FG formulas for immersions of 2D-soliton surfaces.

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