Opinion Formation Games with Dynamic Social Influences

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Opinion formation is a sociological process by which an individual, possibly starting from her innate viewpoint, shapes her belief on a certain subject as a result of the interaction with others (*social influence*).

Several interesting models have been proposed in the literature to assess this phenomenon. In the seminal *DeGroot model* [2], each individual i has an opinion z_i, lying on a real line, which is iteratively updated to the average of the opinions expressed by her acquaintances, e.g., neighbors in a social network. Subsequent models, as the *HK model* by Hegselmann and Krause [5] and the *DW model* by Weisbuch et al. [8], restrict the social influence to only those individuals whose expressed opinion is within a certain distance to z_i (the *confidence region* of individual i). The *FJ model* by Friedkin and Johnsen [4] assumes that individual i also has an innate opinion s_i and i's expressed opinion is then updated by counterbalancing the effects of the social influence with the disagreement between z_i and s_i.

All the above models share the common assumption that the social influence each individual has to undergo remains fixed during the whole duration of the process, e.g., the social network is a static graph. This assumption has been relaxed in some recent works [7,6,3,1] which are based on the evidence that opinion formation and friends selection are often **co-evolving processes** in real life. In particular, Holme and Newman [7] consider the *DeGroot model* (and its generalizations) in which at each step a certain individual i is selected and (i) with probability α, i replaces a random individual from her set of acquaintances with a random individual from the set of people whose expressed opinion coincides with z_i; (ii) with probability $1-\alpha$, a random individual in the set of i's acquaintances changes her opinion to z_i. Bhawalkar et al. [1], instead, consider the *FJ model* in which the disagreement with the innate opinion and the social influence are both expressed as individual's specific functions; moreover, for a given positive integer K, they also investigate the variant in which, for each individual i, the set of acquaintances is formed by the K individuals whose expressed opinion is at minimum distance from s_i.

The co-evolutionary opinion formation models of Holme and Newman [7] and Bhawalkar et al. [1] still assume that the underlying social relationships are not completely dynamic, as they only allow for an individual's set of acquaintances to vary over time. Quantitatively speaking, this means that the social influence that an individual exercises on somebody else can only have a dichotomic behavior: it may appear or disappear, but, whenever present, its magnitude remains fixed.

Since real-life social relationships may either strengthen or weaken over time, it is natural to assume that so will also evolve the attitude that an individual may have on influencing a friend's opinion. Moreover, due to homophyly, i.e., the tendency of individuals to associate and bond with similar others, it also happens that an individual's expressed opinion influences in turn the strength of her social relationships. Based on these evidences, Bhawalkar et al. [1] conclude their paper by proposing a general co-evolutionary opinion formation game with dynamic (i.e., opinion-dependent) social relationships.

Bhawalkar et al. [1] only show that their proposed co-evolutionary opinion formation games with dynamic social relationships always admit pure Nash equilibria. To the best of our knowledge, despite the relevance of their paper, no progresses have been done so far on (specializations of) this model. In this work, we try to fill this gap by embarking on the study of a basic, yet interesting class of opinion formation games with dynamic social relationships.

Let $\mathbf{z}$ be the vector containing the expressed opinions of all players, so that z_i is the expressed opinion of player i. We define a cost-minimization n-player game in which the cost incurred by player i in the profile defined by $\mathbf{z}$ is given by

$$c_i(\mathbf{z}) = \frac{\sum_{j \neq i} w_{ij}(\mathbf{z}) \cdot (z_i - z_j)^2}{\sum_{j \neq i} w_{ij}(\mathbf{z})} + \rho \cdot (s_i - z_i)^2,$$

where $w_{ij}(\mathbf{z})$ is the social influence that j exercises over i which, being a function of $\mathbf{z}$, changes dynamically as the game evolves. More particularly, for a fixed $k > 0$, we set $w_{ij}(\mathbf{z}) = (1 - |s_i - z_j|)^k$. As it can be easily seen, the more z_j is close to s_i, the more j influences i's opinion. The first term of $c_i(\mathbf{z})$ is the cost that i incurs for disagreeing with the society and is defined as the average of the quadratic distances of i's expressed opinion from the expressed opinion of the others weighted by their social influences. The second term of $c_i(\mathbf{z})$, instead, is the quadratic distance of i's expressed opinion from her innate one, scaled by the player's stubbornness (we assume that all players have the same stubbornness). The higher ρ, the less a player is willing to deviate from her innate opinion because of the social pressure.

In this work, we focus on the case in which, for each player i, the innate opinion $s_i \in [0, 1]$, while the expressed opinion $z_i \in \{0, 1\}$. Despite their apparently simplicity, these games are able to capture several interesting scenarios. For instance, think of the situation in which one has to decide whether or not to buy a certain product given that she is not yet completely in favor of one of the two alternatives, or of the situation in which one has to choose between two candidates that might not both exactly reflect her own political ideas.

We show that any game in this class always admits an ordinal potential which implies the existence of pure Nash equilibria and convergence of better-response dynamics starting from any arbitrary strategy profile. Moreover, we prove that any pure Nash equilibrium and any social optimum (with respect to the problem of minimizing the sum of the players' costs) share the same structural property: if one numbers the players in non-decreasing order according to their innate opinions, the sequence of expressed opinions is also non-decreasing, i.e., it can be split into two (possibly empty) subsequences such that the first is made up of only zeroes and the second is made up of only ones. As a consequence, one obtains a simple and efficient algorithm for computing the set of pure Nash equilibria and social optima of a given game (since one has to discriminate among $n + 1$ candidate strategy profiles only).

We also focus on the efficiency losses due to selfish behavior and give upper and lower bounds on the price of anarchy and lower bounds on the price of stability that only depend on the players' stubbornness, i.e., they neither involve the variable k nor the number of players n. In particular, we show that the price of anarchy is unbounded for $\rho \in (0, 1]$, while it is between $\left(\frac{\rho+1}{\rho-1}\right)^2$ and $2\left(\frac{\rho+1}{\rho-1}\right)^2$ for $\rho > 1$. For any value of ρ, the lower bound is attained in the situation in which both consensuses (i.e., all players expressing opinion 0, or all players expressing opinion 1) are pure Nash equilibria, but the players reach the wrong one, that is, the one yielding the highest social cost. We conjecture that our lower bound is tight, but proving a matching upper bound seems to be quite a challenging task, perhaps requiring tedious machineries. For such a reason, even if we are able to derive a better result than the above mentioned factor-2 upper bound, we decided to present a simpler (but still intricate) proof in this conference version. For the price of stability, instead, we only have some preliminary results, as we can just show a lower bound of $\frac{\rho^2+6\rho+1}{(\rho+1)^2}$ for the case of $\rho > 1$ (holding even when $n = 2$), and that there is a 5-player game for which the price of stability is greater than one whenever $\rho \in \left(\frac{217}{566}, 1\right]$.

REFERENCES

1. K. Bhawalkar, S. Gollapudi, and K. Munagala. Coevolutionary Opinion Formation Games. In *Proceedings of the 45th ACM Symposium on Theory of Computing (STOC)*, ACM Press, pp. 41–50, 2013.
2. M. H. DeGroot. Reaching a consensus. *Journal of American Statistical Association*, 69:118–121, 1974.
3. R. Durret, J. P. Gleeson, A. L. Lloyd, P. J. Mucha, F Shi, D. Sivakoff, J. E. S. Socolar, and C. Varghese. Graph fission in an evolving voter model. In *Proceedings of National Academy of Sciences*, 109(10):3682–3687, 2012.
4. N. E. Friedkin and E. C. Johnsen. Social influence and opinions. *Journal of Mathematical Sociology*, 15(3-4):193–205, 1990.
5. R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Artificial Societies and Social Simulations*, 5(3), 2002.
6. A. D. Henry, P. Pralat, and C. Q. Zhang. Emergence of segregation in evolving social networks. *Proceedings of the National Academy of Sciences*, 108:8605–8610, 2011.
7. P. Holme and M. E. J. Newman. Nonequilibrium phase transitions in the coevolution of networks and opinions. *Physical Review E*, 74:056108, 2006.

8. G. Weisbuch, G. Deffuant, F. Amblard, and J.-P. Nadal. Meet, discuss and segregate! *Complexity*, 7(3):55–63, 2002.