A Catalog of $\exists\mathbb{R}$-Complete Decision Problems About Nash Equilibria in Multi-Player Games

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The *Existential Theory of the Reals,* denoted as ETR, is the set of existential first-order sentences over the real numbers. In 1948, Alfred Tarski used his method of quantifier elimination [17] to show that the entire *First Order Theory of the Reals,* encompassing ETR, is decidable, albeit without an elementary bound on its complexity. To date the best known upper bound to decide ETR is $\mathcal{PSPACE}$, coming from the seminal work of Canny [4].

Many geometric, graph-drawing and topological problems have been recognized to have the same complexity as ETR. Some of them concern recognizing *intersection graphs* of a certain type — see, e.g., [14,15]; others concern deciding the stretchability of pseudolines [10]: given a family of plane curves, are they homeomorphic to a line arrangement? Based on these, the complexity class $\exists\mathbb{R}$ was defined by Schaefer and Štefankovič [16] as the set of problems with a polynomial-time, many-to-one reduction to ETR. Decision variants of fixed-point problems, including the *Brouwer fixed-point* problem and *Nash equilibria,* were shown $\exists\mathbb{R}$-complete in [16]; Nash equilibrium [11,12] is undoubtedly the most influential solution concept in Game Theory, representing a state of a *game* where no *player* could unilaterally switch her *strategy* to improve her *utility.* Both *search* and *decision problems* about Nash equilibria have been studied extensively in *Algorithmic Game Theory*; by the seminal results in [5,7], their search problem is $\mathcal{PPAD}$-complete [13] even for 2-player games.

More specifically, Schaefer and Štefankovič [16, Corollary 3.5] identified the *first* $\exists\mathbb{R}$-complete decision problem about Nash equilibria in multi-player games: this is $\exists$ NASH IN A BALL, which asks, given an *r-player game* with $r \geq 3$ and a rational ϱ, whether or not it has a Nash equilibrium with no probability exceeding ϱ. The proof employed a reduction from BROUWER, another decision problem shown $\exists\mathbb{R}$-complete in [16], which asks whether or not a function, represented by a given straight-line program, has a fixed point in a specified ball. Very recently, Garg *et al.* [8] used a chain of *problem-specific* reductions, starting from $\exists$ NASH IN A BALL [16], to prove that four among the $\mathcal{NP}$-complete problems for 2-player games [1,6,9] are $\exists\mathbb{R}$-complete for r-player games with $r \geq 3$. Garg *et al.* [8, Appendix H] posed as an open problem the enlargement of the class of such $\exists\mathbb{R}$-complete problems.

A full story is known for decision problems about Nash equilibria for 2-player games; they are $\mathcal{NP}$-complete; see [1,6,9] for an extensive catalog. Their membership in $\mathcal{NP}$ is due to the fact that the Nash equilibria for a 2-player game involve rational probabilities; this allows, given the supports, polynomial time verification of the Nash equilibrium property. This is no longer the case for r-player games with $r \geq 3$, which may have Nash equilibria with irrational probabilities. Hence, these decision problems are only known to be $\mathcal{NP}$-hard over r-player games with $r \geq 3$, and their precise complexity characterization has remained elusive. (Two notable exceptions are the problems of deciding the existence of a *rational* Nash equilibrium [2] and a *uniform* Nash equilibrium [3], which belong to $\mathcal{NP}$ for r-player games with $r \geq 3$, and this finalizes their complexity classification.) *In this work, we show that they are (almost) all $\exists\mathbb{R}$-complete, delivering an extended catalog of $\exists\mathbb{R}$-complete decision problems about Nash equilibria for r-player games with $r \geq 3$.*

We employ a *game reduction* that maps, given an arbitrary number $\delta > 0$, a pair of 3-player games $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, called the *subgames,* to a 3-player game G with a larger set of strategies for each player; both games $\widetilde{\mathsf{G}}$ (with δ added to each utility) and $\widehat{\mathsf{G}}$ are "embedded" in G as subgames. The reduction guarantees certain correspondences between the Nash equilibria for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively, and those for G. Specifically, a Nash equilibrium for G subsumes either a Nash equilibrium for $\widetilde{\mathsf{G}}$ or one for $\widehat{\mathsf{G}}$; in the other direction, a Nash equilibrium for $\widehat{\mathsf{G}}$ always induces one for G; but a Nash equilibrium for $\widetilde{\mathsf{G}}$ induces one for G if and only if none of its probabilities exceeds $\frac{1}{2}$.

We proceed to embed the game reduction into a polynomial time reduction from $\exists$ NASH IN A BALL to a catalog of decision problems about Nash equilibria for r-player games with $r \geq 3$, thus establishing their $\exists\mathbb{R}$-hardness. We are given an instance $\widetilde{\mathsf{G}}$ of $\exists$ NASH IN A BALL, called the *inbox game*. We construct a game $\widehat{\mathsf{G}}$, called the *gadget game*, which may depend on $\widetilde{\mathsf{G}}$. Finally, we apply the game reduction on $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$ to get the game G. The correspondences between the Nash equilibria for $\widetilde{\mathsf{G}}$ and $\widehat{\mathsf{G}}$, respectively, and those for G are used to deduce the properties of the Nash equilibria for G, which are found to depend on whether or not the inbox game $\widetilde{\mathsf{G}}$ is a positive instance for $\exists$ NASH IN A BALL. The established equivalence between $\widetilde{\mathsf{G}}$ being a positive instance for $\exists$ NASH IN A BALL and the induced properties of G imply the $\exists\mathbb{R}$-hardness of the properties.

The *single, unifying reduction* we employ to establish the $\exists\mathbb{R}$-hardness of all decision problems in the catalog is extremely simple, as well as its corresponding proof; thus, it simplifies tremendously the corresponding chain of (player-specific) reductions in [8], which had involved proofs but only yielded *four* $\exists\mathbb{R}$-hard problems, which are encompassed in the catalog we present. The catalog includes (almost) *all* the decision problems about Nash equilibria for 2-player games shown $\mathcal{NP}$-complete in [1,6,9].

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