On the Robustness of the Approximate Price of Anarchy in Generalized Congestion Games

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The celebrated notion of robust price of anarchy introduced by Roughgarden in [3] has lately given rise to much interest in the determination of inefficiency bounds for pure Nash equilibria which may automatically extend to some of their appealing generalizations, such as mixed Nash equilibria, correlated equilibria and coarse correlated equilibria. These three types of solutions have a particular flavor since, differently from pure Nash equilibria, they are always guaranteed to exist by Nash's Theorem; moreover, the last two ones can also be efficiently computed and even easily learned when a game is repeatedly played over time.

To this aim, Roughgarden [3] identifies a class of games, called *smooth games*, for which a simple three-line proof, called *smoothness argument*, shows significant upper bounds on the price of anarchy of pure Nash equilibria as long as the social function measuring the quality of any strategy profile in the game is ***sum-bounded***, that is, upper bounded by the sum of the players' costs. He then defines the *robust price of anarchy* of a smooth game as the best-possible (i.e., the lowest) upper bound which can be derived by making use of this argument and provides an *extension theorem* which shows that, still for sum-bounded social functions, the price of anarchy of coarse correlated equilibria of any smooth game is upper bounded by its robust price of anarchy. Finally, he shows that several games considered in the literature happen to be smooth and that the class of (unweighted) congestion games with non-negative and ***non-decreasing latency functions*** is *tight* for the ***utilitarian social function*** (that is, the social function defined as the sum of the players' costs), in the sense that, in this class of games, the worst-case price of anarchy of pure Nash equilibria exactly matches the robust price of anarchy. This last result has been subsequently extended to the class of weighted congestion games by Bhawalkar, Gairing and Roughgarden in [1].

In this work, we generalize the tightness result by Bhawalkar, Gairing and Roughgarden along the following four directions:

1. the class of games we consider is a broad generalization of that of weighted congestion games. In particular, we focus on *generalized weighted congestion games*, that is, games in which each player's *perceived cost* is defined as a certain linear combination of all the players' *individual costs* originally experienced in some underlying weighted congestion game.
2. the families of social functions we consider are generalizations of both the utilitarian and the egalitarian social functions (where the egalitarian social function is defined as the maximum of the players' costs). In particular, a family of utilitarian social functions is obtained by summing up a certain contribution from each player, whereas a family of egalitarian social functions is obtained by taking the maximum contribution among the players, where each player's contribution is given by a conic combination of the players' individual costs. We stress that such a combination may significantly differ from the one used to define the players' perceived costs, so that there exist social functions in both families that may not be sum-bounded;
3. the latency functions we consider in the definition of the players' individual costs are selected from a family of allowable non-negative functions with no additional restrictions. This permits us to encompass also latency functions not considered so far in the previous tightness results known in the literature, such as, for instance, the widely used fair cost sharing rule induced by the Shapley value [4];
4. the solution concepts we consider are the approximate versions of all the four types of equilibria named so far. In particular, for any real value $\epsilon \geq 0$, we focus on either ϵ-approximate pure Nash equilibria and ϵ-approximate coarse correlated equilibria.

More precisely, but still informally speaking, we prove the following result: *for a variety of utilitarian and egalitarian social functions and for any real value $\epsilon \geq 0$, the worst-case price of anarchy of ϵ-approximate pure Nash equilibria coincides with that of ϵ-approximate coarse correlated equilibria in the class generalized weighted congestion games with non-negative latency functions.*

As it can be appreciated, the above tightness result generalizes the previous one by Bhawalkar, Gairing and Roughgarden along all four directions simultaneously. The technique we use to prove the theorem is the primal-dual method that we introduced in [2]. In fact, as a byproduct of our proof, it also follows that, in the above considered scenario of investigation, *the worst-case price of anarchy of ϵ-approximate pure Nash equilibria can always be determined through the primal-dual method.*

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