Parabolic problems associated to elliptic operators with second order discontinuous coefficients

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We studied parabolic problems associated to the second order elliptic operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - b|x|^{-2}$$

with $a > 0$ and b, c real coefficients. The leading coefficients are uniformly elliptic but discontinuous at 0, if $a \neq 1$, and singularities in the lower order terms appear when b or c is different from 0. The operator commutes with dilations, in the sense that $I_s^{-1} L I_s = s^2 L$, if $I_s u(x) = u(sx)$.

The operator L with $b = c = 0$ has been originally considered to provide counterexamples to elliptic regularity, see [13], [2]. A priori estimates and elliptic solvability when $a \geq 1$, $b = c = 0$ have been successively investigated in [5], [6] in bounded domains and the spectrum has been computed in dimension 2 in [7].

In [9] we gave necessary and sufficient conditions for the validity of Rellich and Calderón-Zygmund inequalities in L^p. In [10], we proved generation results and domain characterization. In a recent paper (see [12]) we focused on upper kernel estimates for the transition kernel.

If $1 < p < \infty$, we defined the maximal operator $L_{p,max}$ through the domain

$$D(L_{p,max}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\}) : Lu \in L^p(\mathbb{R}^N)\}$$

and observed that, by local elliptic regularity, $L_{p,max}$ is closed and

$$D(L_{p,max}) = \{u \in L^p(\mathbb{R}^N) : Lu \in L^p(\mathbb{R}^N) \text{ as a distribution in } \mathbb{R}^N \setminus \{0\}\}.$$

The operator $L_{p,min}$ is defined as the closure, in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\mathbb{R}^N \setminus \{0\})$ (the closure exists since this operator is contained in the closed operator $L_{p,max}$) and it clear that $L_{p,min} \subset L_{p,max}$.

The equation $Lu = 0$ has radial solutions $|x|^{-s_1}$, $|x|^{-s_2}$ where s_1, s_2 are the roots of the indicial equation $f(s) = -s^2 + (N-1+c-a)s + b = 0$ given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D}$$

where

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2.$$

The above numbers are real if and only if $D \geq 0$. When $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f, see [11] and [10]. This constitutes a generalization of a famous result due to Baras and Goldstein, see [1], in the case of the Schrödinger operator with inverse square potential where the above condition reads $b + (N-2)^2/4 \geq 0$. We point out, however, that even when $b+(N-2)^2/4$ is negative there are realizations of the operator L in $L^2(\mathbb{R}^N)$ which generate analytic semigroups. Such semigroups are not positive and these realizations are necessarily non self-adjoint, see [8].

Assuming $D \geq 0$ we have shown in [10] that there exists and intermediate operator $L_{p,min} \subset L_{p,int} \subset L_{p,max}$ which generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2 + 2)$.

Our main result in [12] is that the semigroup generated by $L_{p,int}$ is analytic of angle $\pi/2$ and that the spectrum of $L_{p,int}$ coincide with $(-\infty, 0]$.
To prove that the semigroup extends to the right half plane we showed complex estimates for the heat kernel. We did not use directly the semigroup generated by $L_{p,int}$ but we used form methods to construct the same semigroup in a weighted L^2-space L^2_μ and then implement Davies' trick to get upper estimates involving a Gaussian factor, the eigenfunction $|x|^{-s_1}$ and another power $|x|^\gamma$ which takes into account the fact that the operator is not symmetric in L^2 with respect to the Lebesgue measure. We followed the approach of [3] and [4] but we covered all cases where the semigroup exists, also the critical case $D = 0$ and, moreover, we obtained estimates for complex times. We did not prove lower bounds; however in the case $a = 1$, that is when the leading part reduces to the Laplacian, our upper bounds coincide with those of [3, Example 4.10] and are almost optimal due to the lower bounds of [3, Example 5.4].

Having good kernel estimates we showed that the semigroup constructed in L^2_μ extends to all $L^p(\mathbb{R}^N)$ (with respect to the Lebesgue measure) when $N/p \in (s_1, s_2 + 2)$ and finally that it coincides with the semigroup generated by $L_{p,int}$. As a consequence we deduced that the spectrum of $L_{p,int}$ coincides with the half-line $(-\infty, 0]$.

We also analysed the operator in spaces of continuous functions.

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