Nonlocal interactions and fractional elliptic equations*

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In the papers [11], [10], we have afforded some problems related to fractional powers of elliptic operators, which are known to be tied with *nonlocal* interactions. We start by describing the classical framework. Let s be a parameter in $(0,1)$. Let us define the *Gagliardo seminorm* of $f:\mathbb{R}^n\to\mathbb{R}$

$$[f]^2_{\dot H^s}=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dxdy, \tag{1}$$

the *fractional perimeter* of a Borel set $E\subset\mathbb{R}^n$

$$P_s(E)=\frac{1}{2}[\chi_E]^2_{\dot H^{s/2}}=\int_{E\times E^c}\frac{1}{|x-y|^{n+s}}\,dxdy, \tag{2}$$

where χ_E is the characteristic function of E and $E^c=\mathbb{R}^n\setminus E$, and the *fractional laplacean*

$$(-\Delta)^s=\frac{1}{\Gamma(-s)}\int_0^\infty\left(e^{t\Delta}-\mathrm{Id}\right)\frac{dt}{t^{1+s}},\qquad \Gamma(-s)=-\frac{\Gamma(1-s)}{s}, \tag{3}$$

where $e^{t\Delta}$ is the classical heat semigroup. As usual in functional calculus, definition (3) is motivated by the elementary computation

$$\lambda^s=\frac{1}{\Gamma(-s)}\int_0^\infty\left(e^{-\lambda t}-1\right)\frac{dt}{t^{1+s}},\quad \lambda>0.$$

Another way of defining the classical laplacean is via the Fourier transform, $\widehat{(-\Delta)^s}f=|\xi|^{2s}\hat f$, but the inversion formula (up to a constant depending on n and s) leads formally to a convolution against the singular kernel $|x|^{-(n+2s)}$, the inverse Fourier transform of the symbol $|\xi|^{2s}$.

Our concerns in [11], [10] are an *isoperimetric* problem and a symmetry result for solutions of a class of nonlinear elliptic equations in the *Wiener space*.

The commonest framework for infinite dimensional analysis is that of a separable Banach space X endowed with a Gaussian measure γ, see [2] for a complete introduction to the subject. The outcoming structure is known as *abstract Wiener space* (X,H,γ), i.e., an infinite dimensional separable Banach space X endowed with a Gaussian measure $\gamma=\mathcal{N}(0,Q)$ (centred normal distribution with covariance Q); here H is the *Cameron-Martin space* associated with the measure γ. We refer also to [9] for a quick introduction to the subject. In particular, we consider the Ornstein–Uhlenbeck operator Δ_γ associated with the Dirichlet form

$$\mathcal{E}(u,v)=\int_X[\nabla_\gamma u(x),\nabla_\gamma v(x)]_H\,d\gamma(x).$$

The operator Δ_γ can be written as $\Delta_\gamma=\mathrm{div}_\gamma\nabla_\gamma$. The Ornstein-Uhlenbeck operator allows us to define the *fractional perimeter* of a set $E\subset X$ as

$$P_{\gamma,s}(E)=\frac{1}{2}[\chi_E]^2_{H^{s/2}_\gamma},$$

where $[\chi_E]^2_{H^{s/2}_\gamma}$ is a suitable fractional norm analogous to (1), and the *fractional powers* $(-\Delta_\gamma)^s$ of $-\Delta_\gamma$ as in (3) through the Ornstein-Uhlenbeck semigroup $e^{t\Delta_\gamma}$.

The isoperimetric inequality in abstract Wiener spaces states that halfspaces minimise the perimeter among sets of the same volume. This result was first proved by Borell in [3]. Later, it was shown in [7,4] (see also [1, Remark 4.7]) that halfspaces are the unique minimisers of the perimeter, with a volume constraint.

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The purpose of [11] is to introduce a notion of fractional perimeter in an abstract Wiener space (X, γ, H) and study the symmetry properties of minimisers for this functional. Our main result is that halfspaces are the unique isoperimetric sets for the fractional perimeter, as it happens for the usual perimeter.

Owing to the well-known relation between the isoperimetric problem and the Allen-Cahn energy [8] (see also [6] for an extension of the result to Wiener spaces, and [12] for a nonlocal version in finite dimensions), we also prove the one-dimensional symmetry of minimisers of the corresponding nonlocal Allen-Cahn energy (see Theorem 2).

The main results are the following.

Theorem 1. *For any $s \in (0,1)$ and $m \in (0,1)$ there exists a set $E_m \subset X$ which solves the isoperimetric problem*

$$\min\Big\{P_{\gamma,s}(E) : \ E \subset X, \ \gamma(E) = m\Big\}. \tag{4}$$

Moreover, the set E_m is necessarily a half-space.

Theorem 2. *Let $m > 0$ and $F : \mathbb{R} \to \mathbb{R}$ be lower semicontinuous, and assume that the problem*

$$\min\Big\{[w]_{H^s_{\gamma_1}} + \int_{\mathbb{R}} F(w)\,d\gamma_1 : \ \int_{\mathbb{R}} w\,d\gamma_1 = m\Big\} \tag{5}$$

admits a minimiser, where γ_1 is the standard Gaussian measure on $\mathbb{R}$. Then the unique minimisers of the problem

$$\min\Big\{[u]_{H^s_\gamma} + \int_X F(u)\,d\gamma : \ \int_X u\,d\gamma = m\Big\} \tag{6}$$

are given by $u(x) = \varphi(\hat{h}(x))$ for some minimiser φ of problem (5) and for some $h \in H$.

Let us come to the (related) results in [10]. Here, we study a boundary reaction problem on the space $X \times \mathbb{R}$, where X is an abstract Wiener space, and we prove that smooth bounded solutions enjoy a symmetry property, i.e., are one-dimensional in a suitable sense. As a corollary of our result, we obtain a symmetry property for some solutions of the following equation

$$(-\Delta_\gamma)^s u = f(u) \qquad \text{on } X. \tag{7}$$

with $s \in (0,1)$, where $(-\Delta_\gamma)^s$ denotes a fractional power of the Ornstein-Uhlenbeck operator, and we prove that for any $s \in (0,1)$ monotone solutions are one-dimensional. Indeed, we investigate the following boundary reaction problem

$$\begin{cases} \operatorname{div}_{\gamma,y}(\mu(y)\nabla_{\gamma,y}v(x,y)) = 0 & \text{on } X \times \mathbb{R}^+ \\[2ex] -\lim\limits_{y\to 0^+} \mu(y)\partial_y v(x,y) = f(v) & \text{on } X \end{cases} \tag{8}$$

$\mu : \mathbb{R}^+ := (0, +\infty) \to \mathbb{R}^+$ is a degenerate weight. In the previous equation, $\operatorname{div}_{\gamma,y}$ and $\nabla_{\gamma,y}$ stand for the divergence and gradient operators in $X \times \mathbb{R}^+$.

Owing to the well-known relation between the Bernstein problem and the symmetry properties of solutions of Allen-Cahn type equation, we prove the one-dimensional symmetry of monotone solutions to (7). This is in the spirit of other symmetry results obtained in connection to a conjecture by De Giorgi on the flatness of level sets of entire solutions of the Allen-Cahn equation in the Euclidean space [5].

Our main result is the following theorem.

Theorem 3. *Let $v \in C^1(X \times \mathbb{R}^+) \cap L^\infty(X \times \mathbb{R}^+)$ satisfy (8), where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that*

$$\partial_i\partial_j v \in C(X \times \mathbb{R}^+) \qquad \text{for all } i, j \in \mathbb{N}$$

and for any $y > 0$

$$\inf_{x \in B_R} [\nabla_\gamma v(x,y), w]_H > 0$$

for all $R > 0$ and for some $w \in H$.

Then, v is one-dimensional in x, in the sense that there exist $V : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $\omega \in X^*$ such that

$$v(x,y) = V(\langle x, \omega\rangle, y) \qquad \textit{for all } x \in X, \quad y > 0.$$

Theorem 3 also admits the following consequence.

Theorem 4. *Let $u \in L^\infty(X) \cap C(X)$ be a weak solution of* (7)*, where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that*

$$\inf_{x \in B_R} [\nabla_\gamma u(x), w]_H > 0$$

for all $R > 0$ and for some $w \in H$. Furthermore, assume that its extension v to $X \times \mathbb{R}^+$ defined by

$$v = \inf\Big\{ \int_{X \times \mathbb{R}^+} y^{1-2s} |\nabla_{\gamma,y} w|^2 d\gamma\, dy, w \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}\, dy),\ w(x,0) = u \Big\}$$

satisfies the assumptions of Theorem 3. Then, u is one-dimensional , in the sense that there exist $U : \mathbb{R} \to \mathbb{R}$ and $\omega \in X^*$ such that

$$u(x) = U(\langle x, \omega \rangle) \qquad for\ all\ x \in X.$$

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