L^p-estimates for parabolic systems with unbounded coefficients coupled at zero and first order

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Second order elliptic and parabolic operators with unbounded coefficients have received a great deal of attention because of their analytical interest as well as their applications to stochastic analysis, both in the autonomous and, more recently, in the nonautonomous case. Due to the applications in Stochastics, much of the work has been done in spaces of continuous and bounded functions and in the L^p-spaces with respect to *the invariant measure*, in the autonomous, and *evolution systems of measures*, in the nonautonomous case. The existence of a unique classical solution for homogeneous parabolic Cauchy problems associated with operators with unbounded coefficients in spaces of continuous and bounded functions, or equivalently the existence of a *semigroup* $T(t)$ or an *evolution operator* $G(t,s)$, respectively, can be shown under mild assumptions on the growth of the coefficients.

On the other hand, the analysis in the L^p setting with respect to the Lebesgue measure has an independent analytical interest and it turns out to be much more difficult than the analysis in the space of continuous and bounded functions or in L^p-spaces with respect to the invariant measure (resp. evolution system of measures). Even in the autonomous case, the Cauchy problem may be not well posed in $L^p(\mathbb{R}^d,dx)$ if the coefficients are unbounded, unless they satisfy very restrictive assumptions. For instance, in the 1-dimensional case very simple operators, such as $D^2-|x|^\varepsilon xD$, with $\varepsilon>0$, do not generate any semigroup in $L^p(\mathbb{R},dx)$ and in this situation, the lack of the potential term plays a crucial role.

Since nowadays many of the results obtained concern the single equations, the aim of this paper is the study of parabolic systems with unbounded coefficients, coupled in the zero and first order terms, in the Lebesgue space $L^p(\mathbb{R}^d,\mathbb{R}^m)$. We consider the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x)=(\boldsymbol{\mathcal{A}}(t)\mathbf{u})(t,x), & t>s\in I, x\in\mathbb{R}^d,\\ \mathbf{u}(s,x)=\mathbf{f}(x), & x\in\mathbb{R}^d\end{cases}\tag{1}$$

where I is an open right-halfline or the whole $\mathbb{R}$ and the elliptic operators

$$\boldsymbol{\mathcal{A}}\mathbf{v}=\sum_{i,j=1}^d D_i(q_{ij}D_j\mathbf{v})+\sum_{i=1}^d B_iD_i\mathbf{v}+C\mathbf{v}\tag{2}$$

have unbounded coefficients $q_{ij}:I\times\mathbb{R}^d\to\mathbb{R}$ and $B_i,C:I\times\mathbb{R}^d\to\mathbb{R}^{m^2}$ $(m\geq 1)$.

Second order elliptic and parabolic systems have been already studied in the simplest case of *zero order coupling*, i.e., when $B_i=b_iI_m$ (see [4,3]). The more general frame of *first order coupling*, i.e., uncoupled diffusion and coupled drift and potential, has been very recently studied in the space of continuous and bounded functions in [1], where the existence of an *evolution operator* $\mathbf{G}(t,s)$ associated with $\boldsymbol{\mathcal{A}}(t)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ has been shown. In this paper we take advantage of such construction and of a pointwise estimate shown in [1] to start our investigation on the properties of $\mathbf{G}(t,s)$ in the L^p context.

We assume that the coefficients are regular enough, namely locally $C^{\alpha/2,\alpha}$, for some $\alpha\in(0,1)$, together with the first order spatial derivatives of q_{ij} and of the entries of B_i, for any $i,j=1,\ldots,d$, and that the matrices $Q(t,x)=[q_{ij}(t,x)]_{i,j=1,\ldots,d}$ are uniformly positive definite.

The L^p analysis is carried out under two different sets of assumptions which are independent. The two approaches give slightly different results. Indeed, in one case we deal directly with the vectorial problem. Using the pointwise estimate proved in [1] we prove that the evolution operator $\mathbf{G}(t,s)$ extends to a bounded and strongly continuous operator in $L^p(\mathbb{R}^d;\mathbb{R}^m)$ for any $p\in[1,+\infty)$.

On the other hand, in the other case we estimate $|\mathbf{G}(t,s)\mathbf{f}|^p$ in terms of $G(t,s)|\mathbf{f}|^p$ for any $t>s\in I$, $p\in[p_0,+\infty)$ and some $p_0>1$. Here, $G(t,s)$ is the evolution operator which governs an auxiliary scalar problem. As a consequence of this comparison result, the boundedness of $\mathbf{G}(t,s)$ in $\mathcal{L}(L^p(\mathbb{R}^d;\mathbb{R}^m))$ for $p\in[p_0,+\infty)$, can be obtained as a byproduct of the boundedness of $G(t,s)$ in $\mathcal{L}(L^1(\mathbb{R}^d))$. Sufficient conditions in order that

$G(t,s)$ is bounded in L^p for any $p \in [1,+\infty)$ can be found in [2].

Going further, we find conditions for the hypercontractivity of $\mathbf{G}(t,s)$. More precisely, under suitable assumptions, we prove that

$$\|\mathbf{G}(t,s)\mathbf{f}\|_{L^q(\mathbb{R}^d;\mathbb{R}^m)} \leq c\|\mathbf{f}\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)}, \quad (3)$$

for any $t \in (s,T]$, $T > s \in I$, $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^m)$, $p \leq q$ and some positive constant c depending on p, q, s and T. Actually, in some cases we prove that the same assumptions which guarantee that $L^p(\mathbb{R}^d, \mathbb{R}^m)$ is preserved by the action of $\mathbf{G}(t,s)$, allow us to prove (3) for any $2 \leq p \leq q$. Then, arguing by duality we establish (3) also when $1 \leq p \leq q \leq 2$. Applying this hypercontractivity result to the scalar evolution operator $G(t,s)$ and using the pointwise estimate of $|\mathbf{G}(t,s)\mathbf{f}|^p$ in terms of $G(t,s)|\mathbf{f}|^p$, we provide conditions for (3) to hold for $p_0 \leq p \leq q$.

The hypercontractivity estimate (3), in this generality, seems to be new also in the autonomous scalar case.

Next, we prove some pointwise estimates for the spatial derivatives of $\mathbf{G}(t,s)\mathbf{f}$. Under additional assumptions, which are essentially growth conditions on the coefficients of the operator $\boldsymbol{\mathcal{A}}(t)$ and their derivatives, we show that there exist positive constants c_1, c_2 such that

$$|D_x\mathbf{G}(t,s)\mathbf{f}|^p \leq c_1 G(t,s)(|\mathbf{f}|^p + |D\mathbf{f}|^p) \quad (4)$$

and, under more restrictive conditions, that

$$|D_x\mathbf{G}(t,s)\mathbf{f}|^p \leq c_2(t-s)^{-\frac{p}{2}} G(t,s)|\mathbf{f}|^p, \quad (5)$$

for any $t \in (s,T]$, $T > s \in I$, $\mathbf{f} \in C^1_c(\mathbb{R}^d;\mathbb{R}^m)$ and $p \in [p_1, +\infty)$ for some $p_1 > 1$.

Now, if the scalar evolution operator $G(t,s)$ preserves $L^1(\mathbb{R}^d)$, estimates (4) and (5) yield that the evolution operator $\mathbf{G}(t,s)$ belongs to $\mathcal{L}(W^{1,p}(\mathbb{R}^d;\mathbb{R}^m))$ and to $\mathcal{L}(L^p(\mathbb{R}^d;\mathbb{R}^m), W^{1,p}(\mathbb{R}^d;\mathbb{R}^m))$, respectively. As a consequence of this fact, we show that $\mathbf{G}(t,s)$ is bounded from $W^{\theta_1,p}(\mathbb{R}^d;\mathbb{R}^m)$ into $W^{\theta_2,p}(\mathbb{R}^d;\mathbb{R}^m)$ for any $0 \leq \theta_1 \leq \theta_2 \leq 1$ and any $p \geq p_1$.

We believe that estimates (4) and (5) could represent a helpful tool to study the evolution operator $\mathbf{G}(t,s)$ in L^p-spaces with respect to *a natural extension to the vector case of evolution systems of measures*, whose definition and analysis is deferred to a future paper. Indeed, already in the scalar case, pointwise gradient estimates have been a key tool to study the asymptotic behaviour of the evolution operator associated with the problem and in establishing some summability improving results for such operator in the L^p spaces with respect the tight time dependent family of invariant measures.

The last section of the paper is devoted to exhibit some classes of operators which satisfy our assumptions.

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