Restricted enveloping algebras whose skew and symmetric elements are Lie metabelian

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Let A be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ = \{x \in A |\, x^* = x\}$ the set of symmetric elements of A under $*$ and by $A^- = \{x \in A |\, x^* = -x\}$ the set of skew-symmetric elements. A general question of interest is to establish the extent to which the structure of A^+ or A^- determines the structure of A (see [8]). For instance, a celebrated result of Amitsur in [1] states that if A^+ or A^- satisfies a polynomial identity, then so does A. Moreover, a considerable amount of attention has been devoted to decide if Lie properties satisfied by the symmetric or the skew symmetric elements of a group algebra $\mathbb{F}G$ under the classical involution, induced from the map $g \mapsto g^{-1}$ on G, are also satisfied by the whole algebra $\mathbb{F}G$ (see e.g. [6,10]). Furthermore, similar questions with respect to involutions of $\mathbb{F}G$ obtained as a linear extension of a group involution of G have been considered, for instance, in [5,9,11].

Now, let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of L. We denote by $\top$ the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^\top = -x$ for every x in L. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$. In [14] and [16] the conditions under which $u(L)^-$ or $u(L)^+$ are Lie solvable, Lie nilpotent or bounded Lie Engel were provided. It turns out that $u(L)^-$ or $u(L)^+$ are Lie solvable if and only if so is $u(L)$. The aim of this note is to characterize L when $u(L)^-$ or $u(L)^+$ are Lie metabelian. If S is a subset of L then we denote by $S^{[p]}$ the subspace spanned by the elements $x^{[p]}$, $x \in S$. Moreover, we use the symbol L' for the derived subalgebra of L. Our main result is the following:

Theorem 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following statements hold:*

1) *$u(L)^-$ is Lie metabelian if and only if either L is abelian or $p = 3$, L' is 1-dimensional and central, and ${L'}^{[p]} = 0$.*
2) *$u(L)^+$ is Lie metabelian if and only if one of the following conditions is satisfied:*
   (i) *L is abelian;*
   (ii) *$p = 3$, L' is 1-dimensional and central, and ${L'}^{[p]} = 0$;*
   (iii) *$p = 3$ and L is 2-dimensional.*

Note that Lie metabelian restricted enveloping algebras have been characterized in [15]. By combining this result and our main theorem, one concludes that in odd characteristic $u(L)^-$ is Lie metabelian if and only if so is $u(L)$. This remains true for the symmetric case provided that $p > 3$, but if L is a 2-dimensional non-abelian restricted Lie algebra over a field of characteristic 3, then $u(L)^+$ is Lie metabelian whereas $u(L)$ is not. It seems interesting that this is indeed the only exception. We also show that in characteristic 2 our main theorem fails both for skew and symmetric case. We finally mention that analogous results for group algebras have been carried out in [3,4,12,13].

REFERENCES

1. S.A. Amitsur: *Rings with involution*, Israel J. Math. **6** (1968), 99–106.
2. S.A. Amitsur: *Identities in rings with involutions*, Israel J. Math. **7** (1969), 63–68.
3. O. Broche - A. del Rio - M. Ruiz: *Group rings with Lie metabelian set of symmetric elements*, Forum Math. **27** (2015), 3533–3566.
4. F. Catino - G. Lee - E. Spinelli: *Group algebras whose symmetric elements are Lie metabelian*, Forum Math. **26** (2014), 1459–1471.

5. A. Giambruno - C. Polcino Milies - S.K. Sehgal: *Lie properties of symmetric elements in group rings*, J. Algebra **321** (2009), 890–902.
6. A. Giambruno - S.K. Sehgal: *Lie nilpotence of group rings*, Comm. Alg. **21** (1993), 4253–4261.
7. I.N. Herstein: *Special simple rings with involution*, J. Algebra **6** (1967), 369–375.
8. I.N. Herstein: *Rings with Involution*, University of Chicago Press, Chicago, 1976.
9. E. Jespers - M. Ruiz Marín: *On symmetric elements and symmetric units in group rings*, Comm. Algebra **34** (2006), 727–736.
10. G. Lee -S.K. Sehgal - E. Spinelli, *Group algebras whose symmetric and skew elements are Lie solvable*, Forum Math. **21** (2009) 661–671.
11. G. Lee - S.K. Sehgal - E. Spinelli: *Lie properties of symmetric elements in group rings. II*, J. Pure Appl. Algebra **213** (2009), 1173–1178.
12. G. Lee - E. Spinelli: *Lie metabelian skew elements in group rings*, Glasgow Math. J. **56** (2014), 187–195.
13. F. Levin - G. Rosenberger: *On Lie metabelian group rings*, Results in Math. **26** (1994), 83-88.
14. S. Siciliano: *On the Lie algebra of skew-symmetric elements of an enveloping algebra*, J. Pure Appl. Algebra **215** (2011), 72–76.
15. S. Siciliano - E. Spinelli: *Lie metabelian restricted universal enveloping algebras*, Arch. Math. (Basel) **84** (2005), 398 – 405.
16. S. Siciliano - H. Usefi: *Lie identities on symmetric elements of restricted enveloping algebras*, Israel J. Math. **195**(2013), 999–2012.
17. H. Strade - R. Farnsteiner: *Modular Lie algebras and their representations.* Marcel Dekker, New York, 1988.