Restricted Lie algebras with maximal 0-PIM

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For a fixed prime number p Wolfgang Willems considered the class $\mathcal{P}_0(p)$ of all finite groups G for which the dimension of the projective cover of the one-dimensional trivial $\mathbb{F}G$-module over a field $\mathbb{F}$ of characteristic p has minimal dimension and compared $\mathcal{P}_0(p)$ to the class of all finite groups having a p-complement. In particular, every p-solvable group belongs to $\mathcal{P}_0(p)$, but the converse is not true (see [33, Section 4]). More recently, Gunter Malle and the third author of this paper classified all finite non-abelian simple groups belonging to $\mathcal{P}_0(p)$ for a fixed prime number p by using the classification of finite simple groups (see [17, Theorem A]). As a consequence, they obtained that a finite group G is solvable (i.e., G is p-solvable for every prime number p) if, and only if, $G \in \mathcal{P}_0(p)$ for every prime number p (see [17, Corollary B]).

In this paper we investigate an analogous question for finite-dimensional restricted Lie algebras over a field of prime characteristic. It turns out that for a restricted Lie algebra there is a canonical upper bound for the dimension of the projective cover of its one-dimensional trivial module. We say that a finite-dimensional restricted Lie algebra has *maximal 0-PIM* if this maximal possible dimension is attained (see Section 2). One main goal of the paper is then to classify all finite-dimensional restricted Lie algebras having maximal 0-PIM. We prove that a finite-dimensional restricted Lie algebra over a field of characteristic $p > 3$ has maximal 0-PIM if, and only if, it is solvable. A main ingredient of the proof is the classification of finite-dimensional simple Lie algebras of absolute toral rank two over an algebraically closed field of characteristic $p > 3$ due to Sasha Premet and Helmut Strade (see [19], [20], and [21]). A comprehensive exposition of the classification of finite-dimensional simple Lie algebras of arbitrary absolute toral rank can be found in the volumes of Helmut Strade (see [26], [28], [29]) and also in the survey [22]. Although not being sufficient for our purposes it should be mentioned that the classification of the restricted simple Lie algebras over an algebraically closed field of characteristic $p > 7$ by Richard Block and Robert Wilson (see [1]) was an essential step for the classification result of Sasha Premet and Helmut Strade.

We show by an example that in characteristic 2 there exists a finite-dimensional non-solvable restricted Lie algebra having maximal 0-PIM. In characteristic 3 we do not know of such a counterexample, but according to the lack of a classification of finite-dimensional simple Lie algebras of absolute toral rank two in this case, at the moment it is not clear whether our result holds in characteristic 3.

In the first section we collect several useful results for projective covers of modules over reduced enveloping algebras that will be needed later in the paper. Some of these results were already known in special cases, but for the convenience of the reader we treat them here in one place. The most important one is the Lie-theoretic analogue of a result for finite-dimensional group algebras due to Wolfgang Willems (see [33, Lemma 2.6] or also [15, Lemma VII.14.2]). It will be used in the proof of the main result of the next section, namely, that every finite-dimensional solvable restricted Lie algebra has maximal 0-PIM. In Section 3 this result is employed to establish an upper bound for the number of the isomorphism classes of irreducible modules with a fixed p-character for a solvable restricted Lie algebra. This generalizes the known results for nilpotent (see [24, Satz 6]) and supersolvable restricted Lie algebras (see [11, Theorem 4]). For quite some time the first author has conjectured that this bound holds for *every* finite-dimensional restricted Lie algebra (see also [14, Section 10, Conjecture] for simple Lie algebras of classical type). In the fourth section we prove a general result on the compatibility of induction for any filtered restricted Lie algebra of finite depth and finite height with the associated graded restricted Lie algebra which might be of general interest. In particular, the irreducibility of the induced module for the associated graded restricted Lie algebra implies the irreducibility of the corresponding induced module for the filtered restricted Lie algebra. Section 5 discusses some non-graded Hamiltonian Lie algebras and their representations. Here the irreducibility of certain induced modules is obtained from the known corresponding result for the associated graded restricted Lie algebra. In the last section it is proved that in characteristic $p > 3$ the projective cover of the trivial irreducible L-module is induced from the one-dimensional trivial module of a torus of maximal dimension, only if L is solvable. Moreover, we provide a counterexample to this result in characteristic 2. In characteristic 3 this seems to be open, and we hope to come back to this on another occasion.

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