Hartree-Fock and Random Phase Approximation theories in a many-fermion solvable model

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We present an ideal system of interacting fermions where the solutions of the many-body Schrödinger equation can be obtained without making approximations. These exact solutions are used to test the validity of two many-body effective approaches, the Hartree-Fock (HF) and the Random Phase Approximation (RPA) theories. The description of the ground state done by the effective theories improves with increasing number of particles.

In its original version [1], the Lipkin, Meshkov, and Glick (LMG) model consists of N fermions occupying two energy levels, each of them has an N-fold degeneracy. We indicate with ϵ the energy difference between these two levels. Each level is characterised by a quantum number σ which assume the value $+1$ in the upper level and -1 in the lower one, and by a set p of quantum numbers specifying the particular degenerate states within the same level. Only two-body interactions which scatter pairs of particles between the two levels without changing the value of p are considered. The hamiltonian of this model system is given by

$$\begin{aligned} H &= \epsilon\, K_0 - \frac{V}{2}\left(K_+^2 + K_-^2\right) \\ & -\frac{W}{2}\left(K_+K_- + K_-K_+\right). \end{aligned} \tag{1}$$

The green full lines of Fig. 1 and Fig. 2 show ground state energy values, $E_{\rm GS}$, as function of the interaction V for $N = 2, 3, 4$ (Fig. 1) and $N = 6, 8, 20$ (Fig. 2). In the left panels we show the solutions obtained when $W = 0$, and in the right panels those for $W = V$. The results up to $N = 8$ have been obtained by using the analytical expressions shown in ref.[2], while those for $N = 20$ by performing a numerical diagonalization of the hamiltonian matrix with standard techniques.

The HF method [3–5] is one of the most commonly used approaches to describe the ground state of many-fermions systems. The basic HF equations can be obtained in various manners, we consider the HF approach in its variational formulation. This leads to [2]

$$E_{\rm HF} = -\frac{N}{2}\left\{ \begin{array}{ll} \epsilon + W & \text{(region I)}, \\ \dfrac{\epsilon^2 + (N-1)^2(V+W)^2}{2\,(N-1)(V+W)} + W & \text{(II)}. \end{array}\right.$$

The values of the ground state HF energies as a function of the interaction V are shown by the dash-dotted lines of Fig. 1 and Fig. 2 by dotted lines. For $W = 0$ we observe a remarkable difference with the exact solutions, especially in the region I, where the HF energies are constant. In the transition point between the two regions, at $\epsilon = (N-1)V$, the value of the energy is $-\frac{N}{2}\,\epsilon$. For $W = V$ case, we observe a reasonable agreement of the HF solutions with the exact ones, even in the region I. In this case, at the transition point between the two regions, which is located at $\epsilon = 2(N-1)V$, the value of the HF energy is $-\frac{N\,(2\,N-1)}{4\,(N-1)}\,\epsilon$. In the figures, the thick red points indicates these values.

The second effective theory we consider is the RPA, which was originally formulated to describe the excitations of an electron gas induced by plasma fluctuations [6], and in the following has been widely applied to describe harmonic vibrations of many-fermion systems from atoms to nuclei [7]. The main goal of the RPA theory is the description of the excited states of the system, but the theory is based on an ansatz about the ground state which is more elaborated than that used in HF. In ref. [2], we presented the basic steps required by the RPA theory to obtain an expression of the ground state energy,

$$E_{\rm RPA} = E_{\rm HF} - \omega\,|Y|^2\,, \tag{2}$$

where $\omega = \sqrt{A^2 - |B|^2}$, $|Y|^2 = \frac{A-\omega}{2\,\omega}$,

$$A = \left\{ \begin{array}{ll} \epsilon - (N-1)\,W & \text{(region I,)}, \\ \dfrac{3\,(N-1)^2(V+W)^2 - \epsilon^2}{2\,(N-1)(V+W)} - (N-1)\,W & \text{(II)}, \end{array}\right.$$

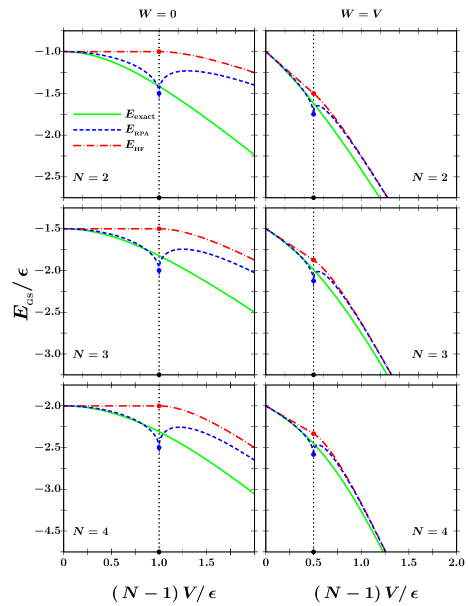

Figure 1. Ground state energies as a function of the interaction strength V for systems composed by $N = 2, 3$ and 4 particles. The full (green) lines indicate the solutions obtained without approximations. The dash-dotted (red) lines the results obtained with the HF model, and the dashed (blue) lines those obtained with the RPA approach. The left panels show the results for $W = 0$, and the right panels those obtained by setting $W = V$. The blue and red thick dots emphasize the values of the HF and RPA energies in the discontinuity line.

and

$$B = \begin{cases} -(N-1)\,V \quad \text{(region I)}, \\ -\dfrac{\epsilon^2 + (N-1)^2(V+W)^2}{2\,(N-1)(V+W)} + (N-1)\,W \quad \text{(II)}. \end{cases}$$

The behaviour of the RPA ground state energies, as a function of the strength V of the interaction is shown in Figs. 1 and 2 by the blue dashed lines. For $W = 0$, it is evident the improvement with respect to the HF results, especially in the region I. The value of the energy at the transition point between the two regions is $-\frac{N+1}{2}\,\epsilon$. For $W = V$ the agreement between RPA and exact results in the region I is excellent. In this case the value of the energy in the transition point is $-\frac{4\,N^2-1}{4\,(N-1)}\,\epsilon$. The behaviour of the solutions for $W = V$ in the region II is remarkable. In this region we find for the RPA solution

$$B = -\frac{\epsilon^2}{4\,(N-1)V}\,. \tag{3}$$

For $4\,(N-1)V \gg \epsilon$ we have that $B \to 0$ and, consequently, due to the fact that $Y \to 0$, the value RPA energy tends to that of the HF energy.

In this article, we tested the validity of the HF and RPA theories in the description of the ground state of the system. In both cases, the solutions are characterised by two regions which depend on the strength of the interaction between the particles. The transition between the two regions is discontinuous. The discontinuity at the meeting point (which seems to suggest some sort of phase transition) is clearly an artefact of the effective theories, since the exact results do not present any discontinuity region.

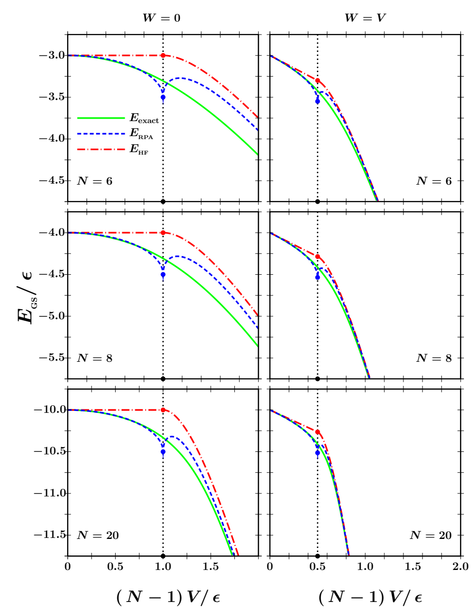

Figure 2. The same as in Fig. 1 for $N = 6, 8$ and 20.

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