A characterization of Sasakian space forms by the spectrum

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Let (M, g) be an n-dimensional compact (connected, smooth and without boundary) Riemannian manifold. We denote by ${}^pSpec(M, g) = \{0 = 0 \leq \lambda_{1,p} \leq \lambda_{2,p} \leq ... \leq \lambda_{k,p} \leq ...+ \uparrow \infty\}$ the spectrum of the Laplace-Beltrami operator acting on the space of p-forms, where each eigenvalue λ_i is written a number of times equal to its multiplicity. The heat equation gives rise to global invariants of the spectrum which describe the asymptotic behavior of the eigenvalues. These global invariants are determined by local invariants of the metric g. Then, a fundamental problem in spectral geometry is to study the relationship between the geometry of (M, g) and its spectrum ${}^pSpec(M, g)$. Many authors studied this problem (see, for example, [?],[?],[?],[?],[?],[?],[?] and the references therein).

Patodi proved the following spectral characterizations of compact Riemannian manifolds.

Theorem A([?]) *Let $(M, g), (M', g')$ be two compact Riemannian manifolds. If ${}^pSpec(M, g) = {}^pSpec(M', g')$, for $p = 0, 1, 2$, then $\dim M = \dim M'$ and:*
(1) (M, g) is an Einstein manifold if and only if (M', g') is so;
(2) (M, g) is of constant sectional curvature c if and only if (M', g') is so.

Regarding the spectral characterizations of Sasakian manifolds, in the recent paper [?] the author shows the following result. Let (M, g, η) and (M', g', η') be two compact Sasakian manifolds, $\dim M = m$, $\dim M' = m'$, $m, m' \geq 5$. If ${}^pSpec(M, g) = {}^pSpec(M', g')$, for $p = 0, 1, 2$, then:
(1) M is η-Einstein if and only if M' is so;
(2) M is a Sasakian space form with constant φ-sectional curvature c if and only if M' is so.

However, in 1981, Olszak proved the following more general result.

Theorem B([?], p.226) *Let (M, g, η) and (M', g', η') be two compact Sasakian manifolds, $\dim M = m$, $\dim M' = m'$, $m, m' \geq 3$. If ${}^pSpec(M, g) = {}^pSpec(M', g')$, only for $p = 0, 1$, then $\dim M = \dim M'$ and :*
(1) M is η-Einstein if and only if M' is so;
(2) M is a Sasakian space form with constant φ-sectional curvature c if and only if M' is so.

In the paper [?] we give a spectral characterization of Sasakian space forms with second Betti number $b_2(M) = 0$, by using only the spectrum ${}^2Spec$. More precisely, the results of our paper [?] are the following.

Theorem 0.1. *Let (M, g, η) and (M', g', η') be two compact Sasakian manifolds, $\dim M = 2n + 1 \neq 15$, $\dim M' = 2n' + 1$, $n, n' \geq 1$. If ${}^2Spec(M, g) = {}^2Spec(M', g')$, then $n = n'$ and M is a Sasakian space form with constant φ-sectional curvature c and second Betti number $b_2(M) = 0$ if and only if M' is so.*

The sphere S^{2n+1} admits a Sasakian structure $(\tilde{\eta}_0, \tilde{g}_0, \tilde{\xi}_0, \tilde{\varphi}_0)$ of space form with constant φ-sectional curvature $c_{\lambda_0} = (4/\lambda_0) - 3$, where λ_0 =const.> 0. We call the Sasakian manifold $(S^{2n+1}, \tilde{\eta}_0, \tilde{g}_0)$ a *Berger-Sasakian sphere* because it is a Sasakian manifold homothetic to a Berger sphere.

Berger metrics are defined as the canonical variation of the standard metric g_0 of constant sectional curvature $+1$ on S^{2n+1}, obtained deforming g_0 along the fibers of the Hopf fibration. In particular we have

Proposition 0.2. *Let (M, η, g) be a compact simply connected η-Einstein Sasakian $(2n+1)$-manifold, and let $(S^{2n+1}, \tilde{\eta}_0, \tilde{g}_0)$ be a Berger-Sasakian sphere with scalar curvature $\tilde{\tau}_0$. If M has scalar curvature $\tau = \tilde{\tau}_0$ and $\mathrm{vol}(M, g) = \mathrm{vol}(S^{2n+1}, \tilde{g}_0)$, then:*
(M, η, g) is isomorphic to $(S^{2n+1}, \tilde{\eta}_0, \tilde{g}_0)$, that is, there is an isometry $f : (M, g) \to (S^{2n+1}, \tilde{g}_0)$ such that $f_*\xi = \tilde{\xi}_0$, $f^*\tilde{\eta}_0 = \eta$ and $f_* \circ \varphi = \tilde{\varphi}_0 \circ f_*$.

Theorem **??** and Proposition **??** imply the following spectral characterization of a Berger-Sasakian sphere.

Theorem 0.3. *Let (M, η, g) be a compact simply connected Sasakian manifold. If ${}^2Spec(M, g) = {}^2Spec(S^{2n+1}, \tilde{g}_0)$, where $\tilde{g}_0$ is a Berger-Sasakian metric, then (M, η, g) is isomorphic to the Berger-Sasakian sphere $(S^{2n+1}, \tilde{\eta}_0, \tilde{g}_0)$.*

Finally, in [**?**] we give some spectral characterization of compact Einstein- Sasakian manifolds in the class of all compact contact Riemannian manifolds. The class of Sasaki-Einstein manifolds is very interesting, its interest comes from Physics. As discussed in the monograph [**?**], Sasaki-Einstein metrics admit Killing spinors, something of great interest to physicists working in conformal quantum field theory. Moreover, any compact simply connected Sasakian space form is a spin manifold ([**?**], p.419). In particular, a Berger-Sasakian sphere is spin.

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