Geometry of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds

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A (connected) pseudo-Riemannian manifold (M, g) is *homogeneous* provided that there exists a group G of isometries acting transitively on it. Such manifold (M, g) can be then identified with $(G/H, g)$, where H is the isotropy group at a fixed point o of M and g is an invariant pseudo-Riemannian metric.

A homogeneous pseudo-Riemannian manifold (M, g) is said to be *reductive* if $M = G/H$ and the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. It is well known that when H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. A fundamental difference arises between homogeneous Riemannian spaces and the pseudo-Riemannian ones. In fact, while any homogeneous Riemannian manifold is reductive, there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition.

Although some differences naturally occur for curvature properties, the study of reductive homogeneous pseudo-Riemannian manifolds parallels the Riemannian case under several points of view. On the other hand, non-reductive examples have not been fully investigated, although it is reasonable to expect that the most interesting differences between Riemannian and pseudo-Riemannian settings occur in such cases.

In this framework, the following topics have been investigated.

Homogeneous geodesics of non-reductive homogeneous pseudo-Riemannian 4-manifolds [?].

Let $(M = G/H, g)$ be a pseudo-Riemannian homogeneous manifold. A geodesic γ through the origin $o \in M = G/H$ is called *homogeneous* if it is the orbit of a one-parameter subgroup. In general, the group G is not unique. If γ is homogeneous with respect to some isometry group G', then it is also homogeneous with respect to the maximal connected group of isometries G, but not conversely.

The study of reductive homogeneous pseudo-Riemannian manifolds parallels the Riemannian case under several points of view, included the study of homogeneous geodesics. In fact, the existence result for homogeneous geodesics in homogeneous Riemannian manifolds extends to the *reductive* pseudo-Riemannian case. Moreover, an algebraic characterization of homogeneous geodesics was obtained for reductive homogeneous spaces of any signature. So, all reductive spaces share the fact that to classify homogeneous geodesics is equivalent to classify geodesic vector fields, of whose homogeneous geodesics are the integral curves, and that such geodesic vector fields are determined by an algebraic condition.

On the other hand, such a characterization fails in the non-reductive case, where different techniques are required. We studied homogeneous geodesics of non-reductive homogeneous pseudo-Riemannian four-manifolds using the completely different approach introduced by Z. Dusek to prove the existence of homogeneus geodesics in homogeneous affine manifolds. We first deduced an explicit description in global coordinates for all invariant metrics of non-reductive homogeneous pseudo-Riemannian four-manifolds. Then, we completely classified Killing vector fields, geodesic vector fields and homogeneous geodesics through a point for these spaces. Several interesting behaviours were found. In particular, in some cases, most of invariant pseudo-Riemannian metrics do not admit any geodesic vector field, although they do admit homogeneous geodesics through a point, according to the fundamental result of Dusek.

Invariant symmetries on non-reductive homogeneous pseudo-Riemannian four-manifolds [?].

Let (M, g) denote a pseudo-Riemannian manifold. For any specified tensor T on (M, g), codifying some mathematical or physical quantity, it is a natural problem to determine the symmetries of T. A *symmetry* is a one-parameter group of diffeomorphisms of (M, g), leaving invariant T. As such, it corresponds to a vector field X, which satisfies $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. Isometries are a well known example of symmetries, for which $T = g$ is the metric tensor and so, X is a Killing vector field. Homotheties and conformal motions on (M, g) provide further examples. In recent years, other symmetries have been investigated, which are related to the curvature of the manifold: *curvature collineations* ($T{=}R$ is the curvature tensor), *Weyl collineations* ($T{=}W$ being the Weyl conformal curvature tensor) and, finally, *Ricci collineations* ($T{=}\varrho$ is the Ricci tensor). Ricci collineations have been considered in several classes of pseudo-Riemannian manifolds, with particular regard to the Lorentzian case because of its physical relevance.

A special interest arised in Lorentzian settings for matter collineations. A *matter collineation* is a vector field X, whose flow preserves the energy-momentum tensor $T = \varrho - \frac{1}{2}\tau g$, where τ denotes the scalar curvature. Besides their mathematical interest as symmetries of T, matter collineations seem to be more relevant from a physical point of view. On the other hand, Ricci collineations have a clear geometrical significance, since ϱ is naturally deduced from the connection of the metric.

Sometimes, one can look for symmetries among vector fields which have a special meaning. In particular, if (M, g) is a homogeneous pseudo-Riemannian manifold, it is natural to study which invariant vector fields on (M, g) preserve a certain tensor. We classified invariant Ricci collineations of all four-dimensional non-reductive homogeneous pseudo-Riemannian spaces, and invariant matter collineations for the Lorentzian cases.

REFERENCES

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