Geometry of paracontact metric structures

G. Calvaruso [1] V. Martin-Molina [2] M.I. Munteanu [3] , A. Perrone [1] and D. Perrone [1]

[1]Dipartimento di Matematica e Fisica, Università del Salento, Italy,

[2]University of Saragoza, Spain,

[3]University of Iasi, Romania

An *almost paracontact structure* on a $(2n+1)$-dimensional (connected) smooth manifold M is a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\varphi(\xi) = 0,\ \eta \circ \varphi = 0,\ \eta(\xi) = 1,\ \varphi^2 = Id - \eta \otimes \xi (1)$$

and the restriction J of φ on the horizontal distribution kerη is an almost paracomplex structure (that is, the eigensubbundles D^+, D^- corresponding to the eigenvalues $1, -1$ of J have equal dimension n). A pseudo-Riemannian metric g on M is *compatible* with the almost paracontact structure (φ, ξ, η) when

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2}$$

In such a case, (φ, ξ, η, g) is said to be an *almost paracontact metric structure*. Remark that, by (1) and (2), $\eta(X) = g(\xi, X)$ for any compatible metric. Any almost paracontact structure admits compatible metrics, which, by (2), necessarily have signature $(n+1, n)$. The *fundamental 2-form* Φ of an almost paracontact metric structure (φ, ξ, η, g) is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for all tangent vector fields X, Y. If $\Phi = d\eta$, then the manifold (M, η, g) (or $(M, \varphi, \xi, \eta, g)$) is called a *paracontact metric manifold* and g the *associated metric*.

Paracontact metric structures were introduced as a natural odd-dimensional counterpart to para-Hermitian structures, just like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been studied in recent years by many authors, emphasizing similarities and differences with respect to the most well known contact case.

In this framework, the following topics have been investigated.

Geometry of H-paracontact metric manifolds [1].

In analogy with the contact metric case, we introduced the definition of H-paracontact metric manifolds, investigating the relationships among the class of H-paracontact spaces and some relevant classes of paracontact metric manifolds. We proved that several classes of paracontact metric manifolds (para-Sasakian and K-paracontact manifolds, paracontact (k, μ)-spaces, three-dimensional homogeneous paracontact metric manifolds) are H-paracontact. Moreover, we proved that all three-dimensional homogeneous paracontact metric manifolds are H-paracontact. Finally, we investigated the case where the Reeb vector field is an infinitesimal harmonic transformation (1-harmonic vector fields) and the paracontact metric structure is a paracontact Ricci soliton. The class of paracontact metric structures whose Reeb vector field is an infinitesimal harmonic transformation is strictly larger than the one of K-paracontact structures (contrarily to the contact metric case).

Paracontact metric structures on the unit tangent sphere bundle [2].

Given a Riemannian manifold M, a so-called natural pseudo-Riemannian metric on the tangent bundle TM has been defined in [M. T. K. Abbassi and M. Sarih, Diff. Geom. Appl. 22 (2005); O. Kowalski and M. Sekizawa, Bull. Tokyo Gakugei Univ. 40 (1988)]. Among others, the Sasaki metrics and the Cheeger-Gromoll metrics are examples of natural metrics. Clearly, a pseudo-Riemannian metric on the unit tangent sphere bundle $T_1M \subset TM$ is also called natural if it is inherited from a natural pseudo-metric on TM. We introduced almost paracontact metric structures on T_1M equipped with natural pseudo-metrics; these structures depend on three real parameters a, b, c. The results obtained connect the Riemannian geometry of the base manifold with properties of these structures. For example, we proved that the following three properties of the structure are equivalent: (i) it is para-Sasakian, (ii) it is K-paracontact, (iii) the base manifold has constant negative sectional curvature and $b = 0$.

Killing magnetic curves in three-dimensional almost paracontact metric manifolds [3].

An important class of magnetic fields, called Killing magnetic fields, are those defined by Killing vector fields. On a 3-dimensional pseudo-Riemannian manifold (M, g), a magnetic field corresponds via the Hodge operator to a closed 2-form F, related to a $(1,1)$-tensor field ϕ, called the Lorentz force, by $g(\phi(X), Y) = F(X, Y)$. A charged particle moving on M under the action of a magnetic field F spirals around a curve, called the magnetic trajectory of F, which satisfies the Lorentz equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \phi(\dot{\gamma})$.

We provided a complete classification of the magnetic trajectories of a Killing characteristic vector field on an arbitrary normal paracontact metric manifold of dimension 3. In the paracontact context the study of magnetic curves is more interesting than its Riemannian analogue, since a metric compatible with a 3-dimensional almost paracontact structure is Lorentzian, and hence the acceleration can have any causal character. Some interesting examples of magnetic curves are obtained on a left-invariant paraSasakian structure of the hyperbolic Heisenberg group and on a paracosymplectic model.

Ricci solitons in three-dimensional paracontact geometry [4].

A pseudo-Riemannian metric g on a smooth manifold M is called a *Ricci soliton* if there exists a a smooth vector field X, such that

$$\mathcal{L}_X g + \varrho = \lambda g, \tag{3}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, ϱ denotes the Ricci tensor and λ is a real number. A Ricci soliton g is said to be a *shrinking*, *steady* or *expanding* according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Ricci solitons are the self-similar solutions of the *Ricci flow* and are important in understanding its singularities. The interest in Ricci solitons has also risen among theoretical physicists in relation with String Theory. After their introduction in the Riemannian case, the study of pseudo-Riemannian Ricci solitons attracted a growing number of authors.

Answering the natural question stated in [1] concerning the existence of nontrivial paracontact Ricci solitons, we completely described paracontact metric three-manifolds whose Reeb vector field satisfies the Ricci soliton equation. While contact Riemannian (or Lorentzian) Ricci solitons are necessarily trivial, that is, K-contact and Einstein, the paracontact metric case allows nontrivial examples. Both homogeneous and inhomogeneous nontrivial three-dimensional examples are explicitly described, and their relationship with paracontact (k, μ)-spaces is clarified.

REFERENCES

1. G. Calvaruso and D. Perrone, *Geometry of H-paracontact metric manifolds*, Publ. Math. Debrecen, 86 (2015), 325346.
2. G. Calvaruso and V. Martin-Molina, *Paracontact metric structures on the unit tangent sphere bundle*, Ann. Mat. Pura Appl., 194 (2015), 1359-1380.
3. G. Calvaruso, M.I. Munteanu and A. Perrone, *Killing magnetic curves in three-dimensional almost paracontact manifolds*, J. Math. Anal. Appl., 426 (2015), 423-439.
4. G. Calvaruso and A. Perrone, *Ricci solitons in three-dimensional paracontact geometry*, J. Geom. Phys., 98 (2015), 1-12.