Nonlinear Schrödinger equations

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1. Polarization interactions in multi-component repulsive Bose-Einstein condensates

Since their first experimental realization, Bose-Einstein condensates (BECs) have attracted considerable attention, and they continue to be the object of intense research. In particular, experiments in multi-component BECs have demonstrated a variety of dark-dark and dark-bright solitons [1,2]. In our work [3,4] we have studied dark-bright soliton interactions in repulsive 3-component BECs, and showed that such interactions result in non-trivial polarization shifts, which are similar in spirit to those in focusing 2-component nonlinear Schrödinger systems [5]. To the best of our knowledge, this is the first time that non-trivial soliton polarization interactions have been reported in a defocusing system.

Repulsive, cigar-shaped single-component BECs can be modeled by the defocusing nonlinear Schrödinger (NLS) equation. Similarly, multi-component BECs are modeled by a vector NLS (VNLS) equation. (In particular, the 2-component case is referred to as the Manakov system [5].) To model dark-bright soliton interactions, one must consider non-zero boundary conditions (NZBC). In our work, we have studied the 3-component defocusing VNLS equation

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2(\|\mathbf{q}\|^2 - q_o^2)\mathbf{q} = \mathbf{0}, \qquad (1)$$

with $\mathbf{q}(x,t) = (q_1, q_2, q_3)^T$ and with the NZBC $\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm = \mathbf{q}_o\, e^{i\theta_\pm}$, with $q_o > 0$ and $\theta_\pm$ arbitrary real constants. The term proportional to q_o in Eq. (1) makes $\mathbf{q}_\pm$ independent of time, but can be removed by a simple gauge transformation.

Equation (1) is a completely integrable system, so its initial value problem can be solved by means of an appropriate inverse scattering transform (IST). In [4] we have formulated the IST for the 3-component case, and in [3] we have employed the IST machinery of [4] to study the resulting soliton interactions.

When only one quartet of discrete eigenvalues is present, letting $z_o = v_o + i\eta_o = |z_o|\, e^{i\alpha_o}$, with $|z_o| < q_o$, one obtains a dark-bright soliton solution of the VNLS equation:

$$q_j(x,t) = -ip_{o,j}\, w_o \sin\alpha_o\, e^{i\Phi_n}\, \mathrm{sech}\, S_o$$

for $j = 1, 2$ and

$$q_3(x,t) = q_+ e^{i\varphi_o} (\cos\alpha_o - i\sin\alpha_o \tanh S_o)\,,$$

where $S_n = \eta_n(x - 2v_nt - x_o)$, $\Phi_n = v_nx - (v_n^2 - \eta_n^2)t$, $w_n = \sqrt{q_o^2 - |z_n|^2}$, $\varphi_n = \alpha_n$ and the unit-norm polarization vector for the bright components is simply $\mathbf{p}_n = (p_{1,n}, p_{2,n})^T = \mathbf{c}_n/\|\mathbf{c}_n\|$.

Soliton interactions and polarization shift.

When two quartets Z_1 and Z_2 of discrete eigenvalues are considered, the corresponding solution of VNLS solution is a nonlinear superposition of two dark-bright solitons A long-time asymptotic analysis then shows that, along the direction of S_n, as $t \to \pm\infty$, the solution takes the form of a 1-soliton solution above, but with $\mathbf{p}_n$, x_n and φ_n replaced by $\mathbf{p}_n^\pm$, $x_n^\pm$ and $\varphi_n^\pm$, with all these quantities obtained in terms of the soliton parameters and norming constants. Such a solution is shown in Fig. 1, where a non-trivial polarization shift is evident for the bright components. Explicitly,

$$\mathbf{p}_1^+ = \chi\left[\mathbf{p}_1^- - 2ir^*(z_1^*/z_2)\,\eta_2 w_2^2\,\langle \mathbf{p}_1^-, \mathbf{p}_2^-\rangle^* \mathbf{p}_2^-\right],$$
$$\mathbf{p}_2^+ = \chi\left[\mathbf{p}_2^- - 2ir\,(z_2/z_1^*)\,\eta_1 w_1^2\,\langle \mathbf{p}_1^-, \mathbf{p}_2^-\rangle\, \mathbf{p}_1^-\right],$$

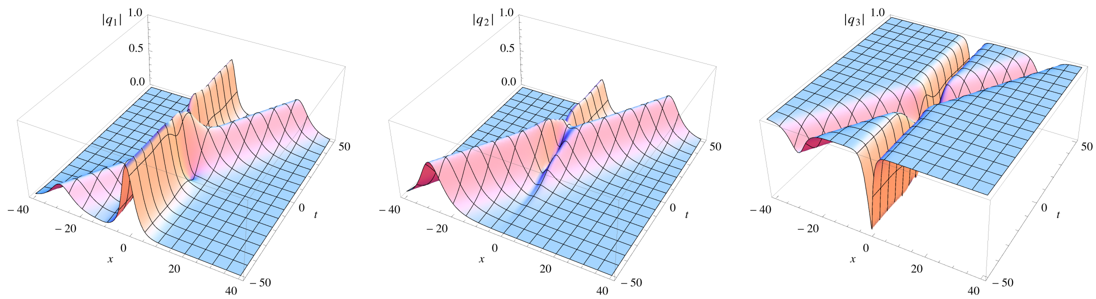

Figure 1. A 2-soliton solution of the defocusing 3-component VNLS equation exhibiting a polarization shift, obtained for $q_o = 1$ with $z_1 = i/2$ (stationary black soliton), $z_2 = (1+i)/4$ (moving gray soliton) and norming vectors $\mathbf{c}_1 = (1,0)^T$ and $\mathbf{c}_2 = (1, 1+i/2)^T$. Note how the bright component of soliton 1 is initially aligned exclusively with q_1, but acquires a component along q_2 as a result of the interaction.

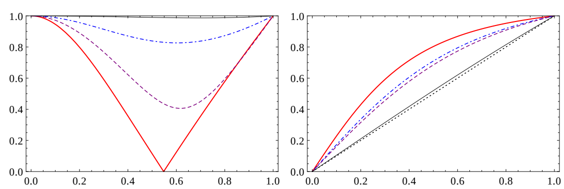

Figure 2. The polarization shift $|\langle \mathbf{p}_1^+, \mathbf{p}_1^- \rangle|$ (left) and output copolarization $|\langle \mathbf{p}_1^+, \mathbf{p}_2^+ \rangle|$ (right) as a function of the input copolarization $|\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle|$ for different combinations of soliton parameters (see [3] for parameter values).

with $r = 1/[(z_1 - z_2)(q_o^2 - z_1 z_2)]$, where $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\dagger \mathbf{b}$ and $\chi = 1/\left[1 + 4\eta_1\eta_2 |r|^2 w_1^2 w_2^2 \, |\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle|^2\right]^{1/2}$.

The product $|\langle \mathbf{p}_1^+, \mathbf{p}_1^- \rangle|$ (quantifying the polarization shift) and the output copolarization $|\langle \mathbf{p}_1^+, \mathbf{p}_2^+ \rangle|$ are shown in Fig. 2 as a function of the input copolarization $|\langle \mathbf{p}_1^-, \mathbf{p}_2^- \rangle|$. The soliton polarizations change unless the soliton polarization vectors are either parallel or orthogonal to each other. Note how the output copolarization is always larger than the input one, even though the interactions occur in a repulsive medium.

The multi-soliton solutions and the corresponding polarization interactions we obtained are stable, therefore we expect the polarization shift to be a robust phenomenon that can in principle be verified experimentally.

2. IST for the focusing NLS equation with a one-sided NZBC

In [6] we have studied the IST as a tool to solve the initial-value problem for the focusing, scalar NLS equation with one-sided non-zero boundary value $q_r(t) \equiv A_r e^{-2iA_r^2 t + i\theta_r}$, $A_r \geq 0$, $0 \leq \theta_r < 2\pi$, as $x \to +\infty$.

The direct problem has been shown to be well-defined for NLS solutions $q(x,t)$ such that $[q(x,t) - q_r(t)\vartheta(x)] \in L^{1,1}(\mathbb{R})$ [here $\vartheta(x)$ denotes the Heaviside function] with respect to $x \in \mathbb{R}$ for all $t \geq 0$, for which analyticity properties of eigenfunctions and scattering data are established.

The inverse scattering problem has been formulated both via (left and right) Marchenko integral equations and as a Riemann-Hilbert problem on a single sheet of the scattering variables $\lambda_r = \sqrt{k^2 + A_r^2}$, where k is the usual complex scattering parameter in the IST.

The direct and inverse problems have also been formulated in terms of a suitable uniformization variable that maps the two-sheeted Riemann surface for k into a single copy of the complex plane. The time evolution of the scattering coefficients has then been derived, showing that, unlike the case of solutions with the same amplitude as $x \to \pm\infty$, in our case both reflection and transmission coefficients have a nontrivial (although explicit) time dependence.

Our results will be instrumental for the investigation of the long-time asymptotic behavior of physically relevant NLS solutions with nontrivial boundary conditions.

REFERENCES

1. C. Becker *et al.*, *Nature Phys.* **4**, 496 (2008)
2. C Hamner *et al.*, *Phys. Rev. Lett.* **106**, 065302 (2011)
3. G. Biondini, D. Kraus, B. Prinari and F. Vitale, *J. Phys. A* **48**, 395202 (2015)
4. B. Prinari, F. Vitale and G. Biondini, *J. Math. Phys.* **56**, 071505 (2015) [selected as featured article for the July 2015 issue of JMP]

5. S. V. Manakov, *Sov. Phys. JETP* **38**, 248 (1974)
6. B. Prinari and F. Vitale, *Cont. Math.* **651** pp. 157-194 (2015)