Structure Preserving Discretizations of the Liouville Equation and their Numerical Tests

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The aim of our research program is to make full use of the theory of Lie groups to study the solution space of discrete equations and in particular to solve difference equations (ΔS), i.e., to difference equations together with the lattice they are written on. The program has two complementary aspects, an analytical and a numerical one. The *analytical aspect* is to determine the maximal symmetry group of the ΔS, i.e., the group of transformations that takes solutions into solutions, and then to use it to obtain exact analytic solutions, at least special ones, if possible general ones. The ΔS to which the approach is applied can come from the study of discrete physical, chemical, biological or other systems, for which symmetries play an important role. Among them we mention phenomena in crystals, or in atomic or molecular chains.

On the other hand ΔS at the scale of the Planck length space-time may very well be discrete. In this case continuous field equations are approximations (continuous limits) of discrete ones. But from the physical point of view very important continuous symmetries should emerge/ be preserved, e.g., when studying quantum field theories on lattices or a fundamental discrete support.

One way to study such a relationship is to preserve symmetries in a discretization of continuous equations, by using symmetry adapted lattices that themselves transform under the group action. This greatly enlarges the set of equations for which symmetry preserving discretization is possible. We will however see that in some cases only a subgroup of the Lie point symmetry group can be preserved as point symmetries.

The *numerical aspect* of our program is the following. When solving an ODE or PDE numerically it is always necessary to replace the continuous equation by a difference system. This can be done in a standard manner, applicable to all equations, simply by replacing derivatives by discrete derivatives. The other possibility takes us directly into the field of geometric integration. The idea is to focus on some important feature of the underlying problem and to preserve it in the discretization. Such a feature may be, for instance linearizability, hamiltonian structure, integrability in the sense of the existence of a Lax pairs and generalized symmetries or point and contact symmetries. We are concentrating on point symmetries and exploring the possibility and usefulness of including them in numerical calculations.

Earlier work has shown that for first-order ODEs preserving a one-dimensional symmetry group provides an exact discretization . For second-order ODEs preserving a 3-dimensional symmetry group often provides analytically solvable schemes .For third- and higher-order ODEs symmetry preserving discretization provides numerical solutions that are, usually, closer to exact ones then those obtained by other methods, specially near to the singularities Quite few has be done for previous work on PDEs

Several recent articles were devoted to discretizations of the Liouville equation [4]

$$z_{xy} = e^{z}, \tag{1}$$

or its algebraic version

$$uu_{xy} - u_x u_y = u^3, \qquad u = e^{z}. \tag{2}$$

The Liouville equation is of interest for many reasons. In differential geometry it is the equation satisfied by the conformal factor $z(x, y)$ of the metric $ds^2 = z^2(dx^2 + dy^2)$ of a two-dimensional space of constant curvature. Then from physical point ov view is a prototype of conformal low-dimensional gravity model. In the theory of infinite-dimensional nonlinear integrable systems it is the prototype of a nonlinear partial differential equation (PDE) linearizable by a transformation of variables, involving the dependent variables (and their first derivatives) alone

$$u = 2\frac{\phi_x \phi_y}{\phi^2}, \qquad \phi_{xy} = 0. \tag{3}$$

In Lie theory this is probably the simplest PDE that has an infinite-dimensional Lie point symmetry group.The symmetry algebra of the algebraic

Liouville equationis given by the vector fields

$$X(f(x)) = f(x)\partial_x - f_x(x)u\partial_u, \tag{4}$$
$$Y(g(y)) = g(y)\partial_y - g_y(y)u\partial_u, \tag{5}$$

where $f(x)$ and $g(y)$ are arbitrary smooth functions.

Equation (4) is a standard realization of the direct product of two centerless Virasoro algebras and we shall denote the corresponding Lie group $\mathrm{VIR}(x) \otimes \mathrm{VIR}(y)$. Restricting $f(x)$ and $g(y)$ to second-order polynomials we obtain the maximal finite-dimensional subalgebra $\mathfrak{sl}_x(2,\mathbb{R}) \bigoplus \mathfrak{sl}_y(2,\mathbb{R})$ and the corresponding finite-dimensional subgroup $\mathrm{SL}_x(2,\mathbb{R}) \otimes \mathrm{SL}_y(2,\mathbb{R})$ of the symmetry group.

The Liouville equation is also an excellent tool for testing numerical methods for solving PDE's, since equationprovides a very large class of exact analytic solutions, obtained by putting

$$\phi(x, y) = \phi_1(x) + \phi_2(y), \tag{6}$$

where $\phi_1(x)$ and $\phi_2(x)$ are arbitrary $\mathbb{C}^{(2)}(I)$ functions on some interval I.

In [1] Adler and Startsev presented a discrete Liouville equation that preserves the property of being linearizable and exactly solvable. In [2] Rebelo and Valiquette wrote a discrete Liouville equation that has the same infinite-dimensional $\mathrm{VIR}(x) \otimes \mathrm{VIR}(y)$ symmetry group as the continuous Liouville equation. The transformations are however generalized symmetries, rather than point ones. In our article [3] we presented a discretization on a four-point stencil that preserves the maximal finite-dimensional subgroup of the $\mathrm{VIR}(x) \otimes \mathrm{VIR}(y)$ group as point symmetries. It was also shown that it is not possible to conserve the entire infinite-dimensional Lie group of the Liouville equation as *point* symmetries. In [3] we also compared numerical solutions obtained using standard (non invariant) discretizations, the Rebelo–Valiquette invariant discretization [2] and our discretization with exact solutions (for 3 different specific solutions). It turned out that the discretization based on preserving the maximal subgroup of point transformations always gave the most accurate results for the considered solutions (all of them strictly positive in the area of integration).

In [4] we explore and compare the different discretizations of the Liouville equation from two points of view. One is a theoretical one, namely to investigate the degree to which different discretizations preserve the qualitative feature of the equation: its exact linearizability, its infinite-dimensional Lie point symmetry algebra, the behavior of the zeroes of the solutions. The other point of view is that of geometric integration: what are the advantages and disadvantages of the different discretizations as tools for obtaining numerical solutions.

In particular we reproduce our previous [3] $\mathrm{SL}_x(2,\mathbb{R}) \otimes \mathrm{SL}_y(2,\mathbb{R})$ symmetry preserving discretization using a 4-point stencil and show that after a slight modification it can reproduce solutions that have horizontal or vertical lines of zeroes (or both). Then, we propose an alternative discretization, using a 9-point stencil, instead of the 4-point one. It approximates the continuous Liouville equation with ϵ^2 precision, as opposed to the ϵ precision of the 4-point discretization. We show that increasing the number of points does not allow us to preserve the entire infinite-dimensional symmetry algebra, nor to treat the lines of zeroes of solutions in a satisfactory manner. Further we take a specific exact solution of the continuous algebraic Liouville equation (2) and approximate it on a 9-point lattice by a numerical solution. The Adler–Startsev discretization is written in a form suitable for numerical calculations and we proved that it possess a different class of generalized symmetries. We did several numerical tests of the invariant 4-point scheme. Five different exact solution of the algebraic Liouville equation are presented and then used to calculate boundary conditions on two lines parallel to the x and y coordinate axes, respectively. The solutions are then calculated numerically using four different discretizations. We compare the validity of the different methods and their qualitative features.

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REFERENCES

1. Adler V.E., Startsev S.Ya., On discrete analogues of the Liouville equation, *Theoret. and Math. Phys.* **121** (1999), 1484–1495,
2. Rebelo R., Valiquette F., Invariant discretization of partial differential equations admitting infinite-dimensional symmetry groups, *J. Difference Equ. Appl.* **21** (2015), 285–318,
3. Levi D., Martina L., Winternitz P., Lie-point symmetries of the discrete Liouville equation,*J. Phys. A: Math. Theor.* **48** (2015), 025204
4. D Levi, **L Martina** and P Winternitz:" Structure Preserving Discretizations of the Liouville Equation and their Numerical Tests", SIGMA 11 (2015), 080, 20 pages