On Stackelberg Strategies in Affine Congestion Games

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Congestion games are, perhaps, the most famous class of non-cooperative games due to their capability to model several interesting competitive scenarios (such as selfish routing, facility location, machine scheduling), while maintaining some nice properties. In these games, there is a set of players sharing a set of *resources*, where each resource has an associated *latency (or cost) function* which depends on the number of players using it (the so-called *congestion*). Each player has an available set of strategies, where each strategy is a non-empty subset of resources, and aims at choosing a strategy minimizing her cost which is defined as the sum of the latencies experienced on all the selected resources.

Congestion games have been introduced by Rosenthal [3]. He proved that each such a game admits a bounded *potential function* whose set of local minima coincides with the set of *pure Nash equilibria* of the game, that is, strategy profiles in which no player can decrease her cost by unilaterally changing her strategic choice. This existence result makes congestion games particularly appealing especially in all those applications in which pure Nash equilibria are elected as the ideal solution concept.

In these contexts, the study of the performance of pure Nash equilibria, usually measured by the *social welfare*, that is, the sum of the costs experienced by all players, has affirmed as a fervent research direction. To this aim, the notion of *price of anarchy* (Koutsoupias and Papadimitriou [2]) is widely adopted. It compares the social welfare of the worst pure Nash equilibrium with that of an optimal solution, called the *social optimum*, that could be potentially enforced by a dictatorial authority. Several recent works have given bounds on the price of anarchy of different variants of congestion games in which the resource latency functions are polynomially bounded in their congestion.

An interesting intermediate situation, usually referred to as *Stackelberg games*, happens when a central authority, called the *leader*, is granted the power of dictating the strategies of a subset of players. The leader's purpose is to determine a good *Stackelberg strategy*, which is an algorithm that carefully chooses the subset of players (called *coordinated players*) and their assigned strategies, so as to mitigate as much as possible the effects caused by the selfish behavior of the *uncoordinated players*, that is, to lower as much as possible the price of anarchy of the resulting game.

The case of our interest, that is (atomic) congestion games with affine latency functions, has been considered before in the literature by Fotakis [1]. For the price of anarchy of the Stackelberg strategy $\mathsf{Largest\ Latency\ First}$, he gives an upper bound of $\min\{(20-11\alpha)/8, (3-2\alpha+\sqrt{5-4\alpha})/2\}$ and a lower bound of $5(2-\alpha)/(4+\alpha)$ with the latter holding even for the restricted case of symmetric strategies, i.e., the case in which all players share the same strategic space. He then considers a randomized variant of Scale (since the deterministic one may be infeasible in the realm of atomic games) and shows that the expected price of anarchy is upper bounded by $\max\{(5-3\alpha)/2, (5-4\alpha)/(3-2\alpha)\}$ and lower bounded by $2/(1+\alpha)$ with the latter holding even for the restricted case of symmetric strategies. He also introduces the Stackelberg strategy λ-Cover which assigns to every resource either at least λ or as many coordinated players as the resource has in the social optimum. For the price of anarchy of this strategy, he proves an upper bound of $(4\lambda-1)/(3\lambda-1)$ for affine latency functions and an upper bound of $1+1/(2\lambda)$ for linear latency functions. Finally, he also gives upper bounds for strategies obtained by combining λ-Cover with either $\mathsf{Largest\ Latency\ First}$ or Scale and upper bounds for games played on parallel links.

Our Contribution. We reconsider the three Stackelberg strategies studied by Fotakis in [1] and give either exact or improved bounds on their price of anarchy. In particular, we achieve the following results: for $\mathsf{Largest\ Latency\ First}$, we show that the price of anarchy is exactly $(20-11\alpha)/8$ for $\alpha \in [0, 4/7]$ and $(4-3\alpha+\sqrt{4\alpha-3\alpha^2})/2$ for $\alpha \in [4/7, 1]$; for λ-Cover, we show that the price of anarchy is exactly $\frac{4\lambda-1}{3\lambda-1}$ for affine latency functions and exactly $1+(4\lambda+1)/(4\lambda(2\lambda+1))$ for linear ones; finally, for Scale, we give an improved upper bound of $1+((1-\alpha)(2h+1))/((1-\alpha)h^2+\alpha h+1)$, where h is the unique integer such that $\alpha \in [(2h^2-3)/(2(h^2-1)), (2h^2+4h-1)/(2h(h+2))]$.

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