Non-uniqueness for second order elliptic operators

G.Metafune [a], M. Sobajima [b], C.Spina [a]

[a]Dipartimento di Matematica e Fisica, Università del Salento, Italy

[b]Tokyo University of Science

In [3], we studied uniqueness and non-uniqueness of realizations of certain singular elliptic operators which generate an analytic semigroup in $L^p(\mathbb{R}^N)$. We described a general method to construct different realizations of abstract operators generating strongly continuous semigroups. More specifically, we consider densely defined operators L_0, $L_{\min}$ and $L_{\max}$ in a Banach space X satisfying $L_{\min} \subsetneq L_0 \subsetneq L_{\max}$ and $\rho(L_0) \neq \varnothing$ and we showed how to construct infinitely many realizations between $L_{\min}$ and $L_{\max}$ different from L_0, with non-empty resolvent set. In some cases we showed that these realizations generate positive or analytic semigroups. We point out that this uniqueness/non-uniqueness problem is different from the classical core problem. If L_0 generates a semigroup $(T(t))_{t\geq 0}$, then $L_{\min}$ is a core for L_0 if and only if $(T(t))_{t\geq 0}$ is the only semigroup whose generator extends $L_{\min}$, see [1, Theorem 1.33]. However, when $L_{\min}$ is not a core for an extension generating a semigroup need to be contained in $L_{\max}$. The proof of [1, Theorem 1.33] in fact shows that these extensions are constructed by preserving the domain of L_0 but changing the operator on $D(L_0) \setminus D(L_{\min})$.

This is not our point of view. Our interest was in differential operators formally given by

$$Lu(x) = \sum_{j,k=1}^{N} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{N} c_j(x)\frac{\partial u}{\partial x_j}(x) - b(x)u(x), \quad x \in \mathbb{R}^N,$$

where a_{jk} $(j,k=1,\dots,N)$, c_j $(j=1,\dots,N)$ and b are real-valued functions which can be singular at 0 and ∞.

The minimal realization $L_{\min}$ of L is defined by considering the closure of the operator on smooth functions with compact support in $\mathbb{R}^N \setminus \{0\}$ and the maximal realization $L_{\max}$ is defined by considering all $u \in L^p(\mathbb{R}^N)$ such that Lu, as a distribution in $\mathbb{R}^N \setminus \{0\}$, belongs to $L^p(\mathbb{R}^N)$. Assuming that a suitable realization $L_{\min} \subset L_0 \subset L_{\max}$ generates a semigroup, we investigated when the same property is shared by other realizations of L between $L_{\min}$ and $L_{\max}$. This means that we changed the domain keeping fixed the expression of the operator on a sufficiently large class of functions. We showed that in some cases of recent interest uniqueness fails also in the class of generators of positive and analytic semigroups. It is clear that this kind of uniqueness never holds for uniformly elliptic operators in bounded domains, since different realizations can be constructed just changing the boundary conditions. Uniqueness, instead, holds in the whole space, if we define $L_{\min}, L_{\max}$ referring to functions and distributions in $\mathbb{R}^N$ instead of $\mathbb{R}^N \setminus \{0\}$, since the generator $(L, W^{2,p}(\mathbb{R}^N))$ coincide both with the minimal and maximal operator. However the situation changes if we consider functions and distributions in $\mathbb{R}^N \setminus \{0\}$ even for the Laplacian, which seems to be artificial (see however [2, Chapter 4] and also Section 3 in this paper) or for singular operators like $\Delta - \frac{b}{|x|^2}$ in the whole space, due to the singularity at 0, or for operators whose coefficients are unbounded at infinity. In these cases, uniqueness in $L^p(\mathbb{R}^N)$ depends on the boundary behavior of Bessel functions and is sensitive to the parameter p.

Observe that sometimes it is easy to construct different L^2-extensions generating a semigroup, by using form methods. However, the positivity of the generated semigroup is not clear by this approach. With our method we showed that, where uniqueness fails, one can construct infinitely many positive and analytic semigroups whose generator is between $L_{\min}$ and $L_{\max}$, and this can be done in L^p. This is the main motivation for the abstract Section 2.

In particular we studied this kind of uniqueness for

$$L = \Delta - \frac{b}{|x|^2} \quad \text{in} \mathbb{R}^N \quad (b \in \mathbb{R});$$

$$L = \Delta + \sum_{j,k=}^{N}(a-1)\frac{x_j x_k}{|x|^2}\frac{\partial^2}{\partial x_j \partial x_k} + \frac{cx}{|x|^2} - \frac{b}{|x|^2} \quad \text{in} \mathbb{R}^N \quad (a>0, c \in \mathbb{R}, b \in \mathbb{R});$$

$$L = |x|^\alpha \Delta + c|x|^{\alpha-2} x \cdot \nabla - b|x|^{\alpha-2} \quad \text{in} \mathbb{R} \quad (\alpha \in \mathbb{R} \setminus \{2\}, c \in \mathbb{R}, b \in \mathbb{R}).$$

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