Elliptic operators with second order discontinuous coefficients

G.Metafune [a], M. Sobajima [b], C.Spina [a]

[a]Dipartimento di Matematica e Fisica, Università del Salento, Italy

[b]Tokyo University of Science

We considered the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - b|x|^{-2},$$

under the condition $a > 0$, which is equivalent to the ellipticity of the principal part, and with $b, c \in \mathbb{R}$. The operator L with $b = c = 0$ has been originally considered to provide counterexamples to elliptic regularity, see [15], [5]. A priori estimates and elliptic solvability when $a \geq 1$, $b = c = 0$ have been successively investigated in [6], [7] in bounded domains and the spectrum has been computed in dimension 2 in [8].

In [11] we gave necessary and sufficient conditions for the validity of Rellich and Calderón-Zygmund inequalities in L^p. Rellich inequalities for the Laplacian in L^2 spaces (according to our notations $a = 1,\ b = c = 0$)

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} |x|^{-4}|u|^2\,dx \leq \int_{\mathbb{R}^N} |\Delta u|^2\,dx,$$

for $N \neq 2$ and for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, have been proved by Rellich in 1956 and then extended to L^p-norms for $1 < p < \frac{N}{2}$ in [14].

Rellich inequalities with respect to the weight $|x|^\alpha$ have been also studied in [3] and [13]. We refer to [4] and [2] for necessary and sufficient conditions in L^2 and to [10] for a general treatment in L^p, valid also for the operators

$$\Delta + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2}.$$

Rellich and Calderón-Zygmund inequalities for the operator L without lower order terms and under the assumption $a \geq 1$ have been proved in [7]. Sharp results concerning Calderón-Zygmund estimates have been obtained in the unit ball of $\mathbb{R}^N$. Some of these estimates have been proved also in the whole space but under suitable restrictions.

We used the same approach of [10] to rewrite Rellich inequalities for L as spectral inequalities for the operator $L(|x|^2\cdot)$ which are easier to treat, since the radial and the angular part commute. Then we used Rellich inequalities to extend the Calderón-Zygmund estimates for the Laplacian to L; we obtained necessary and sufficient conditions for the validity of the estimate

$$\int_{\mathbb{R}^N} |D^2 u|^p\,dx \leq C\int_{\mathbb{R}^N} |Lu|^p\,dx,$$

where $u \in W^{2,p}(\mathbb{R}^N)$, $1 < p < \infty$. We point out that we treated the general case $a > 0$ without assuming $a \geq 1$ as in [6], [7] and [8].

In [12], we proved generation results and domain characterization. To state them and explain how they are proved we introduce some notation. If $1 < p < \infty$, we define the maximal operator $L_{p,max}$ through the domain

$$D(L_{p,max}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\}) : Lu \in L^p(\mathbb{R}^N)\}$$

and note that, by local elliptic regularity, L_{max} is closed and

$$D(L_{p,max}) = \{u \in L^p(\mathbb{R}^N) : Lu \in L^p(\mathbb{R}^N) \text{ as a distribution in } \mathbb{R}^N \setminus \{0\}\}.$$

The operator $L_{p,min}$ is defined as the closure, in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\mathbb{R}^N \setminus \{0\})$ (the closure exists since this operator is contained in the closed operator $L_{p,max}$) and it is clear that $L_{p,min} \subset L_{p,max}$.

The equation $Lu = 0$ has radial solutions $|x|^{-s_1}$, $|x|^{-s_2}$ where s_1, s_2 are the roots of the indicial equation $f(s) = -as^2 + (N-1+c-a)s + b = 0$ given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D} \tag{1}$$

where

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2. \tag{2}$$

The above numbers are real if and only if $D \geq 0$. When $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f. This constitutes a generalization of a famous result due to Baras and Goldstein [1] in the case of the Schrödinger operator with inverse square potential, where the above condition reads $b + (N-2)^2/4 \geq 0$. We point out, however, that even when $b + (N-2)^2/4$ is negative there are realizations of the operator L in $L^2(\mathbb{R}^N)$ which generate analytic semigroups. Such semigroups are not positive and these realizations are necessarily non self-adjoint, see [9].

Assuming $D \geq 0$ we showed that there exists and intermediate operator $L_{p,min} \subset L_{p,int} \subset L_{p,max}$ which generates a (analytic) semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2+2)$. An intuitive explanation (for $D > 0$) of this result is the following. If $u(x) = \sum u_j(r) P_j(\omega)$ and $f = \sum f_j(r) P_j(\omega)$ where (P_j) are spherical harmonics, then the equation $\omega^2 u - Lu = f$, Re $\omega > 0$ can be reduced to the infinite system ODE of Bessel type

$$\omega^2 u_j(r) - \left(u_j''(r) + \frac{N-1+c}{r} u_j'(r) - (b+\lambda_n)\frac{u_j(r)}{r^2}\right) = f_j(r) \tag{3}$$

where n is the degree of P_j and $\lambda_n = n^2 + (N-2)n$ are the eigenvalues of the Laplace-Beltrami operator on the sphere S^{N-1}. Each of the above equation has characteristic numbers $s_1^{(n)}, s_2^{(n)}$, defined as in (1), (2) with $b + \lambda_n$ instead of b. The numbers $s_1^{(n)}$ decrease to $-\infty$, whereas $s_2^{(n)}$ increase to $+\infty$. The equations (3) have more regularizing effect as n increases, since the potentials $(-b+\lambda_n)r^{-2}$ become more and more negative and therefore the most critical equation appears for $n = 0$ and corresponds to radial functions. For positive ω, (3) with $n = 0$ and $f_0 = 0$ has two linearly independent solutions $v_{\omega,1}, v_{\omega,2}$ with the following properties: $v_{\omega,1}$ is exponentially increasing at ∞ and behaves like r^{-s_1} as $r \to 0$, $v_{\omega,2}$ is exponentially decreasing at ∞ and behaves like r^{-s_2} as $r \to 0$. Using these function one can construct a Green function as for Sturm-Liouville problems. However, if $N/p \leq s_1$, then neither $v_{\omega,1}$ or $v_{\omega,2}$ belong to $L^p((0,1), r^{N-1}\,dr)$ and equation (3) with $n = 0$ cannot be solved for suitable f_0. If $N/p \geq s_2 + 2$, the function $v_{\omega,2}$ belongs to the domain of the minimal operator $L_{p,min}$ and is therefore an eigenfunction of any of its extensions. These facts explain the negative part of our result.

If $N/p \in (s_1, s_2)$, then $v_{\omega,1}$ is the only solution of the homogeneous equation which is in L^p near 0 and $v_{\omega,2}$ is the only solution of the homogeneous equation which is in L^p near ∞ (in both cases with respect to the measure $r^{N-1}\,dr$). This means that there is only one way to construct a resolvent and hence $L_{p,max}$ is a generator. By duality, $L_{p,min}$ is a generator when $N/p \in (s_1+2, s_2+2)$. Therefore $L_{p,int} = L_{p,max}$ if $N/p \in (s_1, s_2]$ and $L_{p,int} = L_{p,min}$ if $N/p \in [s_1+2, s_2+2)$ and $L_{p,int}$ is the unique realization of L between $L_{p,min}$ and $L_{p,max}$ which generates a semigroup, when these two intervals overlap, that is when $s_1 + 2 \leq s_2$, since it coincides either with $L_{p,min}$ or with $L_{p,max}$ (and with both when $N/p \in [s_1+2, s_2]$). However, if $s_2 < s_1 + 2$ and N/p is in between, that is when

$$\frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2 \in [0,1) \text{ and } \frac{N}{p} \in (s_2, s_1+2),$$

both functions $v_{\omega,1}, v_{\omega,2}$ are in $L^p((0,1), r^{N-1}\,dr)$ and there is no uniqueness even among the generators of positive and analytic semigroups, see [12]. The choice of the domain of $L_{p,int}$ is made to preserve the coherence of the semigroup in the L^q-scale, by extrapolating the semigroup from those $L^q(\mathbb{R}^N)$ for which there is uniqueness; namely we select $v_{\omega,1}$ to construct the Green function near 0 but other choices are possible.

The above arguments can be made rigorous in L^2 by expansion in spherical harmonics, but not directly in L^p. Instead we use a global argument based on improved Hardy and Poincaré inequalities which yield complex dissipativity on subspaces of $L^p(\mathbb{R}^N)$ generated by high order spherical harmonics and then we perform a one dimensional analysis on a finite number of cases.

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