On the sectoriality of a class of degenerate elliptic operators arising in population genetics

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In the paper [2] we deal with the class of degenerate second order differential operators

$$\mathcal{L}_d u(x) = \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i} u(x), \qquad x \in S^d, \tag{1}$$

where S^d is the canonical simplex of $\mathbb{R}^d$ consisting of all $x \in [0,1]^d$ such that $x_i \geq 0$, for $i = 0, \dots, d$ (here, $x_0 := 1 - \sum_{i=1}^d x_i$), $b := (b_i)_{i=1}^d \subset C(S^d)$ and $\langle b(x), \nu(x)\rangle \geq 0$, for every $x \in \partial S^d$, denoting by ν the unit inward normal at ∂S^d. The operator (1) appears in the theory of Fleming–Viot processes as a generator of a Markov C_0–semigroup defined on $C(S^d)$. Fleming–Viot processes are measure–valued processes that can be viewed as diffusion approximations of empirical processes associated with some classes of discrete time Markov chains in population genetics.

The operator (1) has been largely studied using an analytic approach by several authors in different settings. The interest comes from the fact that the equations describing the diffusion processes are of degenerate type and hence, the classical techniques for the study of (parabolic) elliptic operators on smooth domains cannot be applied. Indeed, the operator (1) degenerates on the boundary ∂S^d of S^d and the domain S^d is not smooth as its boundary presents edges and corners.

The study of such type of degenerate (parabolic) elliptic problems on $C([0,1])$ started in the fifties with the papers by Feller [9,10]. In the Feller theory for the one–dimensional case, it is shown that the behaviour of the diffusion process on the boundary constitutes one of its main characteristics. So, the appropriate setting for studying the equations describing the diffusion process is the space of continuous functions on the simplex S^d.

The subsequent work of Clément and Timmermans [6] clarified which conditions on the coefficients of the operator (1), ensure the generation of a C_0–semigroup in $C([0,1])$. The problem of the regularity of the generated semigroup in $C([0,1])$ has been considered by several authors. In particular, Metafune [11] established the analyticity of the semigroup under suitable conditions on the coefficients of the operator (1).

In the d–dimensional case, Ethier [7] (see also [8, p.375]) proved the existence of a C_0–semigroup of positive contractions on $C(S^d)$ under mild assumption on the drift terms b_i, i.e., the coefficients b_i, $i = 1, \dots, d$, were assumed to be Lipschitz functions on S^d. In [3] the authors proved that if $b = 0$ on S^d then the closure in $C(S^d)$ of the operator $(\mathcal{L}_d, C^2(S^d))$ generates a bounded analytic compact C_0-semigroup of positive contractions and of angle $\pi/2$ on $C(S^d)$.

The aim of the paper [2] is to extend the analyticity's result in the case that the drift b is not identically 0 on S^d. More precisely, the result will be achieved under the assumption that, setting $b_0 := -\sum_{i=1}^d b_i$, for every $i = 0, 1, \dots, d$, the drift coefficient b_i is continuous on S^d and satisfies a suitable Hölder condition with Hölder exponent 1/2 in a neighbourhood of the set $\{x \in S^d \mid x_i = 0\}$. In particular, the result implies generation of a C_0–semigroup of positive contractions on $C(S^d)$ under this condition, thereby improving the result of Ethier. Moreover, gradient estimates for the resolvent map of $\mathcal{L}_d$ are also proved.

The results will be first proved in the particular case of the Shimakura operator

$$\mathcal{L}^\omega_d u(x) = \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial_{x_i x_j} u(x) + \sum_{i=1}^{d} (\omega_i - |\widetilde{\omega}| x_i)\partial_{x_i} u(x), \qquad x \in S^d, \tag{2}$$

where $\widetilde{\omega} = (\omega_0, \omega_1, \dots, \omega_d) \in [0, +\infty[^{d+1}$ and $|\widetilde{\omega}| := \sum_{i=0}^d \omega_i$. This operator was first widely studied by Shimakura in [12,13] (see also [14]). A more general version of (2), with each ω_i strictly positive continuous function on S^d, for $i = 0, \dots, d$, was also studied in [5,1,4]. In particular, in [1] the authors proved that the C_0–semigroup generated by the closure of $(\mathcal{L}^\omega_d, C^2(S^d))$ in $C(S^d)$ is differentiable.

The more general results will be obtained via a perturbation argument.

An appendix is included in the paper [2], where we collect some general results about perturbations and interpolation estimates which have been applied in the proofs of this paper and could be of interest by themself.

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