Analytic semigroups and some degenerate evolution equations defined on domains with corners

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In the paper [1] we deal with the class of degenerate second order elliptic differential operators

$$L = \Gamma(x)\sum_{i=1}^{d}[\gamma_i(x_i)x_i\partial^2_{x_i} + b_i(x)\partial_{x_i}], \quad x \in Q^d = [0,M]^d, \tag{1}$$

where $M > 0$, Γ, b_i and γ_i, for $i = 1,\dots,d$, are continuous functions on Q^d and on $[0,M]$ respectively and $b = (b_1,\dots,b_d)$ is an inward pointing drift. The operator (1) arises in the theory of Fleming–Viot processes, namely measure–valued processes that can be viewed as diffusion approximations of empirical processes associated with some classes of discrete time Markov chains in population genetics. From the analytic point of view, the interest in the operator (1) relies on the fact that it is of degenerate type and its domain presents edges and corners, hence, the classical techniques for the study of (parabolic) elliptic operators on smooth domains cannot be applied.

In the one-dimensional case, the study of such type of degenerate (parabolic) elliptic problems on $C([0,1])$ started in the fifties with the papers by Feller [6,7], where it is pointed out that the behaviour on the boundary of the diffusion process associated with the degenerate operator constitutes one of its main characteristics. The subsequent work of Clément and Timmermans [5] clarified which conditions on the coefficients of the operator (1) guarantee the generation of a C_0–semigroup in $C([0,1])$. In particular, Metafune [8] established the analyticity of the semigroup under suitable conditions on the coefficients of the operator, obtaining, among other results, the analyticity of the semigroup generated by $x(1-x)D^2$ on $C([0,1])$, which was a problem left open for a long time. The latter result was extended to the multidimensional case in [2], where the authors proved the analyticity of the semigroup generated by the operator

$$Au(x) = \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j}u(x)$$

on $C(S^d)$, where S^d is the d-dimensional canonical simplex.

In [4] Cerrai and Clément established Schauder estimates for (1) under suitable Hölder continuity hypothesis on its coefficients.

The aim of the paper [1] is to present some results about generation, sectoriality and gradient estimates for the resolvent of a suitable realization of (1) in $C(Q^d)$.

To this end, in [1] one starts with the analysis in the particular case that the functions b_i are constant and $\Gamma = 1$, first in the one-dimensional case and then, via a tensor product argument, in the multi-dimensional setting. Much attention is paid to the constants appearing in the analyticity and gradient estimates, showing their uniformity for b_i belonging to an interval $[0,B]$. These results strongly rely on estimates proved in [3]. In particular, in [1] it is shown that

Proposition 0.1. *Let $B > 0$ and $\gamma = (\gamma_1,\gamma_2,\dots,\gamma_d) \in (C([0,M]))^d$, with each γ_i strictly positive. Let*

$$\mathcal{L}^{\gamma,b} = \sum_{i=1}^{d}\gamma_i(x_i)\partial^2_{x_i} + b_i\partial_{x_i}.$$

Then, for every $b \in [0,B]^d$, the following properties hold.

1. *There exist $K,\alpha,\bar{t} > 0$ depending on B and on γ such that, for every $u \in C(Q^d)$ and $i = 1,\dots,d$, we have*

$$||t\mathcal{L}^{\gamma,b}T(t)|| \le Ke^{\alpha t}, \quad t \ge 0. \tag{2}$$

$$||\sqrt{x_i}\partial_{x_i}(T(t)u)||_\infty \le \frac{Ke^{\alpha t}}{\sqrt{t}}||u||_\infty, \quad 0 < t < \bar{t}. \tag{3}$$

$$||\sqrt{x_i}\partial_{x_i}(T(t)u)||_\infty \le Ke^{\alpha t}||u||_\infty, \qquad t \ge \bar{t}. \tag{4}$$

Moreover, for every $i \in \{1, \dots, d\}$ *and* $u \in C(Q^d)$, $\sqrt{x_i}\partial_{x_i}(T(t)u) \in C(Q^d)$ *and*

$$\lim_{x_i \to 0^+} \sup_{x_j \in [0,M], j \in \{1,\dots,d\}\setminus\{i\}} \sqrt{x_i}\partial_{x_i}(T(t)u) = 0. \tag{5}$$

2. *There exist* $d_1, d_2, R > 0$ *depending on* B *and on* γ *such that, for every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda > R$, $u \in C(Q^d)$ *and* $i = 1, \dots, d$, *we have*

$$||R(\lambda, \mathcal{L}^{\gamma,b})u||_\infty \le d_1 \frac{||u||_\infty}{|\lambda|}, \tag{6}$$

$$||\sqrt{x_i}\partial_{x_i}(R(\lambda, \mathcal{L}^{\gamma,b})u)||_\infty \le d_2 \frac{||u||_\infty}{\sqrt{|\lambda|}}. \tag{7}$$

Moreover, for every $i \in \{1, \dots, d\}$ *and* $u \in C(Q^d)$, $\sqrt{x_i}\partial_{x_i}(R(\lambda, \mathcal{L}^{\gamma,b})u) \in C(Q^d)$ *and*

$$\lim_{x_i \to 0^+} \sup_{x_j \in [0,M], j \in \{1,\dots,d\}\setminus\{i\}} \sqrt{x_i}\partial_{x_i}(R(\lambda, \mathcal{L}^{\gamma,b})u)(x) = 0. \tag{8}$$

3. *There exist* $C, D, \bar{\varepsilon} > 0$ *depending on* B *and on* γ *such that, for every* $0 < \varepsilon < \bar{\varepsilon}$, $i = 1, \dots, d$ *and* $u \in D(\mathcal{L}^{\gamma,b})$, *we have*

$$\|\sqrt{x_i}\partial_{x_i}u\|_\infty \le \frac{C}{\varepsilon}\|u\|_\infty + D\varepsilon\|\mathcal{L}^{\gamma,b}u\|_\infty.$$

As a consequence, the case of non-constant drift with a perturbation argument is completely determined under the assumption that there exists $\delta > 0$ and $C > 0$ such that, for every $i = 1, \dots, d$ and $x,\ x' \in Q^d$ with $x_i < \delta$ and $x'_i = 0$, we have

$|b_i(x) - b_i(x')| \le C\sqrt{x_i}$,

Actually, in [1] it is shown that

Theorem 0.2. *The closure* $(\mathcal{L}, D(\mathcal{L}))$ *of the operator* $(L, C^2_\diamond(Q^d))$ *generates an analytic compact* C_0*-semigroup* $(T(t))_{t\ge0}$ *of positive contractions in* $C(Q^d)$. *Moreover, all the estimates in Proposition 0.1 hold for the operator* $(\mathcal{L}, D(\mathcal{L}))$.

Finally, in [1] the case that Γ is not a constant function is studied, by applying a "freezing coefficients" proof. An important role in this argument will be played by the uniformity of the constants in the resolvent estimates.

As a by-product of the previous results analogous results are given in [1] for the operator

$$\Gamma(x)\sum_{i=1}^d [\gamma_i(x_i)x_i(1-x_i)\partial^2_{x_i} + b_i(x)\partial_{x_i}], \quad x \in [0,1]^d.$$

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