Infinite dimensional analysis*

D. Pallara [a]

[a]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Italy

The commonest framework for infinite dimensional analysis is that of a separable Banach space X endowed with a Gaussian measure γ, see [11] for a complete introduction to the subject. The outcoming structure is known as *abstract Wiener space* $(X, H, \gamma)$and it finds applications in various fields in Mathematics and Physics, such as mathematical finance, statistical mechanics and hydrodynamics and the path approach to quantum theory or stationary phase estimation in stochastic oscillatory integrals with quadratic phase function. The case of a Hilbert space (which gives rise to a different structure) is presented in [13], [14]. In general, solutions of SDEs are not continuous (and sometimes not even everywhere defined) functionals, hence the notion of weak derivative and Sobolev functional comes into play. Looking at weak differentiation and the study of the behaviour of stochastic processes in domains leads immediately to the need for a good comprehension of integration by parts formulae, something that in the Euclidean case has been completely understood in the frameworks of geometric measure theory, sets with finite perimeter and more generally functions of bounded variation, see [5]. This approach in Wiener spaces has been considered by Fukushima in [15] and Fukushima-Hino in [16], where the first definition of BV functions in infinite dimensional spaces has been given, most likely inspired by a stochastic characterisation of finite perimeter sets in finite dimension already given by Fukushima. The theory has been reformulated in an integralgeometric approach in [6–8] and, in the quoted Hilbert space setting, in [1] and, with a non-Gaussian measure, in [2]. The most important points in this theory have been summarised in form of lecture notes in [19]. The study of BV functions and perimeters has been pursued in two different directions. On the one hand, looking at local fine properties of BV functions: in this vein, the extension to the infinite-dimensional setting of the notions of *approximate limits*, *jump set* and others, see [5] for the classical Euclidean case, are far from being obvious. Some progresses in this directions are contained in [8]. On the other hand, in [18] we deal with BV functions (and sets with finite perimeter) on *convex open subsets* of abstract Wiener spaces (X, H, γ), a theory that is still at its very beginning. We propose a definition of BV functions on domains in X through an integration by parts formula against a suitable class of test functions, rather than merely as restrictions of BV functions on the whole space. This is not a trivial issue because of the lack of smooth bump functions (for general X) and of bounded extension operators. Moreover, we relate the variation of a function to the short time behaviour of the Ornstein-Uhlenbeck semigroup; besides the interest of extending similar results available in different contexts, see [20,10,12,17], such a relation has proved to be useful to describe fine properties of BV functions in Wiener spaces, see [3,4]. In particular, we get a characterisation of BV functions on a convex open set Ω similar to that on the whole space. We consider the Ornstein–Uhlenbeck operator L associated with the Dirichlet form

$$\mathscr{E}(u, v) = \int_\Omega [\nabla_H u(x), \nabla_H v(x)]_H d\gamma(x), \quad u, v \in W^{1,2}(\Omega, \mu),$$

and the semigroup $(T_t)_{t\geq 0}$ generated by the realisation of L in $L^2(\Omega, \gamma)$. The main result of this paper is the next Theorem.

Theorem 1. *Let $\Omega \subset X$ be an open convex set, and let $u_0 \in BV(X, \gamma) \cap L^2(X, \gamma)$ be such that $|D_\gamma u_0|(\partial\Omega) = 0$. Then, for any $t > 0$,*

$$\int_\Omega |\nabla_H T_t u_0(x)|_H d\gamma(x) \leq |D_\gamma u_0|(\Omega)$$

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and

$$\lim_{t\to 0}\int_\Omega |\nabla_H T_t u_0(x)|_H d\gamma(x) = |D_\gamma u_0|(\Omega). \tag{1}$$

The study of the Ornstein–Uhlenbeck semigroup on domains is less straightforward than in the whole X, since no explicit formula is available for $T(t)$; nevertheless, the function $t \mapsto \int_\Omega |\nabla_H T_t u(x)|_H d\gamma(x)$ is still monotone. In the proof of monotonicity the convexity of Ω plays an essential role.

REFERENCES

1. L. Ambrosio, G. Da Prato, D. Pallara: *BV* functions in a Hilbert space with respect to a Gaussian measure, *Rend. Acc. Lincei* **21** (2010), 405-414.
2. L. Ambrosio, G. Da Prato, B. Goldys, D. Pallara: Bounded variation with respect to a log-concave measure, *Comm. P.D.E.* **37** (2012), 2272-2290.
3. L. Ambrosio, A. Figalli: Surface measure and convergence of the Ornstein-Uhlenbeck semigroup in Wiener spaces, *Ann. Fac. Sci. Toulouse Math.* **20** (2011) 407-438.
4. L. Ambrosio, A. Figalli, E. Runa: On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **24** 2013, 111-122.
5. L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford University Press, 2000.
6. L. Ambrosio, S. Maniglia, M. Miranda, D. Pallara: Towards a theory of *BV* functions in abstract Wiener spaces, *Evolution Equations: a special issue of Physica D* **239** (2010), 1458-1469.
7. L. Ambrosio, S. Maniglia, M. Miranda Jr, D. Pallara: *BV* functions in abstract Wiener spaces, *J. Funct. Anal.,* **258** (2010), 785-813.
8. L. Ambrosio, M. Miranda Jr, D. Pallara: Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability, *Discrete Contin. Dyn. Syst.,* **28** (2010), 591-606.
9. L. Ambrosio, M. Miranda, D. Pallara: Some fine properties of *BV* functions on Wiener spaces, *Anal. Geom. Metr. Spaces* **3** (2015), 212-230.
10. L. Angiuli, M. Miranda (Jr), D. Pallara, F. Paronetto: *BV* functions and parabolic initial boundary value problems on domains, *Ann. Mat. Pura Appl* **(4)188** (2009), 297-311.
11. V. I. Bogachev:*Gaussian Measures,* American Mathematical Society, 1998.
12. M. Bramanti, M. Miranda, D. Pallara: Two characterization of *BV* functions on Carnot groups via the heat semigroup, *Int. Math. Res. Not.* **17** (2012), 3845-3876.
13. G. Da Prato:*An introduction to infinite-dimensional analysis,* Springer, 2006.
14. G. Da Prato, J. Zabczyk:Second order partial differential equations in Hilbert spaces, London Mathematical Society Lecture Note Series, 293, Cambridge U. P., 2002.
15. M. Fukushima: *BV* functions and distorted Ornstein-Uhlenbeck processes over the abstract Wiener space, *J. Funct. Anal.* **174** (2000), 227-249
16. M. Fukushima, M. Hino: On the space of BV functions and a Related Stochastic Calculus in Infinite Dimensions, *J. Funct. Anal.* **183** (2001), 245-268
17. B. Güneysu, D. Pallara: Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below, *Math. Ann.* **363** (2015), 1307-1331.
18. A. Lunardi, M. Miranda, D. Pallara: *BV* functions on convex domains in Wiener spaces, *Potential Anal.* **43** (2015), 23-48.
19. M.Miranda, M. Novaga, D. Pallara: An introduction to *BV* functions in Wiener spaces, in: Variational Methods for Evolving Objects, L. Ambrosio, Y. Giga, P. Rybka, Y. Tonegawa eds., *Advanced Studies in Pure Mathematics* **67** (2015), 245-293.
20. M. Miranda (Jr), D. Pallara, F. Paronetto, M. Preunkert: Short-time Heat Flow and Functions of Bounded Variation in $\mathbb{R}^N$, *Ann. Fac. Sci. Toulouse Math.* **16** (2007), 125-145.