# Components of $V(\rho) \otimes V(\rho)$

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Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. We fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ and let $\rho$ be the half sum of positive roots, where the roots of $\mathfrak{b}$ are called the positive roots. For any dominant integral weight $\lambda \in \mathfrak{t}^*$, let $V(\lambda)$ be the corresponding irreducible representation of $\mathfrak{g}$. B. Kostant initiated (and popularized) the study of the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$. In fact, he conjectured the following.

**Conjecture 1.** *(Kostant) Let $\lambda$ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat-Chevalley order on the set of weights.*

It is, of course, clear that if $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$, then $\lambda \leq 2\rho$.

One of the main motivations behind Kostant's conjecture was his result that the exterior algebra $\wedge\mathfrak{g}$, as a $\mathfrak{g}$-module under the adjoint action, is isomorphic with $2^r$ copies of $V(\rho) \otimes V(\rho)$, where $r$ is the rank of $\mathfrak{g}$ (cf. [9]). Recall that $\wedge\mathfrak{g}$ is the underlying space of the standard chain complex computing the homology of the Lie algebra $\mathfrak{g}$, which is, of course, an object of immense interest.

**Definition.** An integer $d \geq 1$ is called a *saturation factor* for $\mathfrak{g}$, if for any $(\lambda, \mu, \nu) \in D^3$ such that $\lambda + \mu + \nu$ is in the root lattice and the space of $\mathfrak{g}$-invariants:

$$[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^{\mathfrak{g}} \neq 0$$

for some integer $N > 0$, then

$$[V(d\lambda) \otimes V(d\mu) \otimes V(d\nu)]^{\mathfrak{g}} \neq 0,$$

where $D \subset \mathfrak{t}^*$ is the set of dominant integral weights of $\mathfrak{g}$. Such a $d$ always exists (cf. [10]; Corollary 44]).

Recall that 1 is a saturation factor for $\mathfrak{g} = sl_n$, as proved by Knutson-Tao [8]. By results of Belkale-Kumar [2] (also obtained by Sam [11] and Hong-Shen [5]), $d$ can be taken to be 2 for $\mathfrak{g}$ of types $B_r, C_r$ and $d$ can be taken to be 4 for $\mathfrak{g}$ of type $D_r$ by a result of Sam [11]. As proved by Kapovich-Millson [6], [7], the saturation factors $d$ of $\mathfrak{g}$ of types $G_2, F_4, E_6, E_7, E_8$ can be taken to be 2 (in fact any $d \geq 2$), $144, 36, 144, 3600$ respectively. (For a discussion of saturation factors $d$, see [10], §10.)

Now, the following result (weaker than Conjecture 1) is our main theorem.

**Theorem.** *Let $\lambda$ be a dominant integral weight such that $\lambda \leq 2\rho$. Then, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$, where $d \geq 1$ is any saturation factor for $\mathfrak{g}$. In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

The proof uses a description of the eigencone of $\mathfrak{g}$ in terms of certain inequalities due to Berenstein-Sjamaar coming from the cohomology of the flag varieties associated to $\mathfrak{g}$, a 'non-negativity' result due to Belkale-Kumar and the following Proposition.

**Proposition.** *Let $\lambda \leq 2\rho$ be a dominant integral weight. Then,*

$$\lambda = \rho + \beta,$$

*for some weight $\beta$ of $V(\rho)$.*

An interesting aspect of our work is that we make an essential use of a solution of the eigenvalue problem and saturation results for any $\mathfrak{g}$.

**Remark.** As informed by Papi, Berenstein-Zelevinsky had proved Conjecture 1 (by a different method) for $\mathfrak{g} = sl_n$ (cf. [4], Theorem 6). They also determine in this case when $V(\lambda)$ appears in $V(\rho) \otimes V(\rho)$ with multiplicity one. To our knowledge, Conjecture 1 appears first time in this paper.

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