Regular functions on spherical nilpotent orbits in complex symmetric pairs I: classical non-Hermitian cases

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Let G be a complex connected semisimple algebraic group, and let K be the fixed point subgroup of an algebraic involution θ of G. Recall that K is reductive. It is connected if G is simply-connected.

The Lie algebra $\mathfrak{g}$ of G splits into the sum of eigenspaces of θ,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where the Lie algebra $\mathfrak{k}$ of K is the eigenspace of eigenvalue 1, and $\mathfrak{p}$ is the eigenspace of eigenvalue -1. The adjoint action of G on $\mathfrak{g}$, once restricted to K, leaves $\mathfrak{k}$ and $\mathfrak{p}$ stable.

Therefore, $\mathfrak{p}$ is an interesting K-module, where one may want to study the geometry of the K-orbits. With this aim, one looks at the so-called nilpotent cone $\mathcal{N}_\mathfrak{p} \subset \mathfrak{p}$, which consists of the elements whose K-orbit closure contains the origin. In this case, $\mathcal{N}_\mathfrak{p}$ actually consists of the nilpotent elements of $\mathfrak{g}$ which belong to $\mathfrak{p}$. By a fundamental result of Kostant and Rallis [22], as in the case of the adjoint action of G on $\mathfrak{g}$, there are finitely many nilpotent K-orbits in $\mathfrak{p}$.

Provided K is connected, we restrict our attention to the spherical nilpotent K-orbits in $\mathfrak{p}$. Here spherical means with an open orbit for a Borel subgroup of K, or equivalently with a ring of regular functions which affords a multiplicity-free representation of K. The classification of these orbits is known and due to King [20].

To study these orbits we make use of the technical machinery of spherical varieties. Given a spherical nilpotent K-orbit $\mathcal{O} \subset \mathfrak{p}$, we obtain complete information on the normality (or non-normality) of its closure $\overline{\mathcal{O}}$, and on the K-module structure of the ring of regular functions of its normalization $\widetilde{\mathcal{O}}$ (which is encoded in the *weight semigroup* $\Gamma(\widetilde{\mathcal{O}})$).

In the present paper we assume that: (1) G is of classical type and (2) θ is of non-exceptional type. The other cases will be treated separately. The latter condition is equivalent to ask that K is semisimple, in this case the symmetric space G/K is also called of non-Hermitian type, and $\mathfrak{p}$ is a simple K-module.

Let $G_\mathbb{R}$ be a real form of G with Lie algebra $\mathfrak{g}_\mathbb{R}$ and Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{p}_\mathbb{R}$, so that θ is induced by the corresponding Cartan involution of $G_\mathbb{R}$. Then K is the complexification of a maximal compact subgroup $K_\mathbb{R} \subset G_\mathbb{R}$, and the Kostant-Sekiguchi-Đoković correspondence [15,31] establishes a bijection between the set of the nilpotent $G_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}$ and the set of the nilpotent K-orbits in $\mathfrak{p}$. Let us briefly recall how it works, more details and references can be found in [13].

Every non-zero nilpotent element $e \in \mathfrak{g}_\mathbb{R}$ lies in an $\mathfrak{sl}(2)$-triple $\{h, e, f\} \subset \mathfrak{g}_\mathbb{R}$. Every $\mathfrak{sl}(2)$-triple $\{h, e, f\} \subset \mathfrak{g}_\mathbb{R}$ is conjugate to a *Cayley triple* $\{h', e', f'\} \subset \mathfrak{g}_\mathbb{R}$, that is, an $\mathfrak{sl}(2)$-triple with

$\theta(h') = -h'$, $\theta(e') = -f'$ and $\theta(f') = -e'$. To a Cayley triple in $\mathfrak{g}_{\mathbb{R}}$ one can associate its *Cayley transform*

$$\{h, e, f\} \mapsto \{i(e-f), \tfrac{1}{2}(e+f+ih), \tfrac{1}{2}(e+f-ih)\} :$$

this is a normal triple in $\mathfrak{g}$, that is, an $\mathfrak{sl}(2)$-triple $\{h', e', f'\}$ with $h' \in \mathfrak{k}$ and $e', f' \in \mathfrak{p}$. By [22], any non-zero nilpotent element $e \in \mathfrak{p}$ lies in a normal triple $\{h, e, f\} \subset \mathfrak{g}$, and any two normal triples with the same nilpositive element e are conjugated under K. Then the desired bijective correspondence is constructed as follows: take an adjoint nilpotent orbit $\mathcal{O} \subset \mathfrak{g}_{\mathbb{R}}$, choose an element $e \in \mathcal{O}$ lying in a Cayley triple and take a Cayley triple $\{h, e, f\}$ containing it, take its Cayley transform $\{h', e', f'\}$ and take the nilpotent orbit $Ke' \subset \mathfrak{p}$.

Among the nice geometrical properties of the Kostant-Sekiguchi-Đoković correspondence, we just recall here one result concerning sphericality: the spherical nilpotent K-orbits in $\mathfrak{p}$ correspond to the adjoint nilpotent $G_{\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}$ which are multiplicity free as Hamiltonian $K_{\mathbb{R}}$-spaces [19].

In accordance with the philosophy of the orbit method (see e.g. [1]), the unitary representations of $G_{\mathbb{R}}$ should be parametrized by the (co-)adjoint orbits of $G_{\mathbb{R}}$. In particular one is interested in the so-called unipotent representations of $G_{\mathbb{R}}$, namely those which should be attached to nilpotent orbits. The K-module structure of the ring of regular functions on a nilpotent K-orbit in $\mathfrak{p}$ (which we compute in our spherical case) should give information on the corresponding unitary representation of $G_{\mathbb{R}}$. Unitary representantions that should be attached to the spherical nilpotent K-orbits are studied in [18] (when G is a classical group), [30] (when G is the special linear group) and [32] (when G is the symplectic group).

The normality and the K-module structure of the coordinate ring of the closure of a spherical nilpotent K-orbit in $\mathfrak{p}$ have been studied in several particular cases, with different methods, by Nishiyama [25], [26], by Nishiyama, Ochiai and Zhu [27], and by Binegar [2].

In Appendix A of the present paper we report the list of the spherical nilpotent K-orbits in $\mathfrak{p}$ for all symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of classical non-Hermitian type.

In the classical cases, the adjoint nilpotent orbits in real simple algebras are classified in terms of signed partitions, as explained in [13, Chapter 9]. In the list, every orbit is labelled with its corresponding signed partition.

For every orbit we provide an explicit description of a representative $e \in \mathfrak{p}$, as element of a normal triple $\{h, e, f\}$, and the centralizer of e, which we denote by K_e. All these data can be directly computed using [20] (but we point out a missing case therein).

The first datum which is somewhat new in this work is the Luna spherical system associated with $\mathrm{N}_K(K_e)$, the normalizer of K_e in K, which is a wonderful subgroup of K. It is equal to $K_{[e]}$, the stabilizer of the line through e, and notice that $K_{[e]}/K_e \cong \mathbb{C}^\times$.

The Luna spherical systems are used to deduce the normality or non-normality of the K-orbits, and to compute the K-modules of regular functions.

Appendix B of the present paper consists of two sets of tables. The first set contains our results on the normality of the spherical nilpotent K-orbits in $\mathfrak{p}$ and on the K-module structure of their rings of regular functions. The second set contains the Luna spherical systems.

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