# The isometry group for the Hamming distance

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Let $X$ be a non-void set, let $V \doteq X^n$ and denote by $d : V \times V \longrightarrow \mathbb{N}$ the Hamming distance (see [1] and, in general, [2]). For $u = (x_1, x_2, \ldots, x_n)$, $v = (y_1, y_2, \ldots, y_n)$ in $V$, this is defined as

$$d(u, v) = |\{1 \leq i \leq n \,|\, x_i \neq y_i\}|$$

So $d(u, v)$ is the number of different coordinates in $u$ and $v$. A bijective map $f : V \longrightarrow V$ is an *isometry* if $d(f(u), f(v)) = d(u, v)$ for all pairs $u, v \in V$. It is clear that the set $\mathsf{O}(V)$ of isometries is a group; the aim of this short note is the description of the structure of such group. This is probably well known in literature.

We denote by $\mathsf{S}_n$ the group of permutations on $n$ elements and by $\mathsf{S}(X)$ the group of permutations of the set $X$. Notice that $\mathsf{S}_n$ acts by automorphisms on the group $\mathsf{S}(X)^n$: $\sigma \in \mathsf{S}_n$ maps $(\tau_1, \tau_2, \ldots, \tau_n)$ to ${}^\sigma\tau \doteq (\tau_{\sigma^{-1}(1)}, \tau_{\sigma^{-1}(2)}, \ldots, \tau_{\sigma^{-1}(n)})$.

For any $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathsf{S}(X)^n$ and $\sigma \in \mathsf{S}_n$ we may define an isometry $f_{\tau,\sigma}$ of $V$ by mapping $v \in V$ to $f_{\tau,\sigma}(v) \doteq (\tau_1 x_{\sigma^{-1}(1)}, \tau_2 x_{\sigma^{-1}(2)}, \ldots, \tau_n x_{\sigma^{-1}(n)})$.

Moreover these isometries are all the isometries of $V$ with respect to the Hamming distance.

**Theorem.** *The map* $\mathsf{S}(X)^N \rtimes \mathsf{S}_n \ni (\tau, \sigma) \longmapsto f_{\tau,\sigma} \in \mathsf{O}(V)$ *is a group isomorphism.*

In the particolar case of $X$ a field we have at once

**Corollary.** *Suppose that $X = \mathbb{F}$ is a field, so that $V$ is an $n$–dimensional vector space over $\mathbb{F}$. Then the group of $\mathbb{F}$–linear isometries of $V$ is isomorphic to the group of (invertible) monomial $n \times n$ matrices.*

The proof of our theorem relies on a simple combinatorial lemma: an isometry is completely determinated by its action on the union of the “axes”. In order to make a clear statement out of this vague assertion we introduce some notation.

Let 0 be a fixed element of $X$. We denote an element $v = (x_1, x_2, \ldots, x_n)$ of $V$ as

$$v = \sum_{i=1}^{n} x_i e_i$$

If a coordinate $x_i$ is 0 then we omit it in the above sum; hence 0 is the element $(0, 0, \ldots, 0)$ of $V$. Notice that, with this notation, we have $f_{\tau,\sigma}(\sum x_i e_i) = \sum \tau_{\sigma(i)}(x_i) e_{\sigma(i)}$. We call the subset of $\{1, 2, \ldots, n\}$ of indeces $i$ such that $x_i \neq 0$, the *support* of $v$ and we denote it by $\mathrm{supp}(v)$; it is clear that $|\mathrm{supp}(v)| = d(v, 0)$.

If $x \in X$ then $xe_i$ is the element of $V$ whose all coordinates are 0 but the $i$–th that is $x$ and, clearly, $\mathrm{supp}(xe_i) = \{i\}$. In particular $Xe_i$ is the set of all elements of $V$ whose all coordinates but the $i$–th are 0; this is the $i$–th *axis* in $V$. Let $A \doteq \cup_{i=1}^{n} Xe_i$ be the union of all axes. Our key lemma is the following

**Lemma.** *If an isometry $f : V \longrightarrow V$ is the identity on $A$ then it is the identity on $V$.*

From this lemma the proof of the theorem follows quite easily.

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## REFERENCES

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