Engel condition on enveloping algebras of Lie superalgebras

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Recall that an associative ring R is said to satisfy the Engel condition if R satisfies the identity

$$[\dots[[x,\underbrace{y],y],\dots,y}_{n}] = 0,$$

for some n. It follows from Zel'manov's celebrated result about the restricted Burnside problem [19] that every finitely generated Lie ring satisfying the Engel condition is nilpotent. Kemer in [5] proved that if R is an associative algebra over a field of characteristic zero that satifies the Engel condition then R is Lie nilpotent. This result was later proved by Zel'manov in [18] for all Lie algebras. However these results fail in positive characteristic, see [17,11]. Nevertheless, Shalev in [13] proved that every finitely generated associative algebra over a field of characteristic $p > 0$ satisfying the Engel condition is Lie nilpotent. This result was further strengthened by Riley and Wilson in [10] by proving that if R is a d-generated associative C-algebra, where C is a commutative ring, satisfying the Engel condition of degree n, then R is upper Lie nilpotent of class bounded by a function that depends only on d and n. Hence, in the positive characteristic case one would need to assume that R is also finitely generated.

Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field $\mathbb{F}$ of characteristic $p \neq 2$ with bracket $(\,,)$. The adjoint map of $x \in L$ is denoted by $\mathrm{ad}x$. We denote the enveloping algebra of L by $U(L)$. In case $p = 3$ we add the condition $((y, y), y) = 0$, for every $y \in L_1$. This identity is necessary to embed L in $U(L)$.

The Lie bracket of $U(L)$ is denoted by $[a, b] = ab - ba$, for every $a, b \in U(L)$. We are interested to know when $U(L)$ satisfies the Engel condition. Note that the Engel condition is a non-matrix identity, that is a polynomial identity not satisfied by the algebra $M_2(\mathbb{F})$ of 2×2 matrices over $\mathbb{F}$. The conditions for which $U(L)$ satisfies a non-matrix identity are given in [2]. It follows from Zel'manov's Theorem [18] that over a field of characteristic zero $U(L)$ satisfies the Engel condition if and only if $U(L)$ is Lie nilpotent. The characterization of L when $U(L)$ is Lie nilpotent over any field of characteristic not 2 is given in [2]. Hence, we have

Corollary. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field of characteristic zero. The following conditions are equivalent:*

1. *$U(L)$ is Lie nilpotent;*
2. *$U(L)$ is bounded Lie Engel;*
3. *L_0 is abelian, L is nilpotent, (L, L) is finite-dimensional, and either $(L_1, L_1) = 0$ or $\dim L_1 \leq 1$ and $(L_0, L_1) = 0$.*

However this result is no longer true in positive characteristic as our following theorem shows.

Theorem 1. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field of characteristic $p \geq 3$. The following conditions are equivalent:*

1. *$U(L)$ is bounded Lie Engel;*
2. *$U(L)$ is PI, L_0 is abelian, $\mathrm{ad}x$ is nilpotent for every $x \in L_0$, and either $(L_1, L_1) = 0$ or $\dim L_1 \leq 1$ and $(L_0, L_1) = 0$;*
3. *$U(L)$ is PI, L_0 is abelian, L is nilpotent, and either $(L_1, L_1) = 0$ or $\dim L_1 \leq 1$ and $(L_0, L_1) = 0$.*

Note that the above theorem does not follow from Zel'manov or Riley and Wilson's results because $U(L)$ is not necessarily finitely generated.

Now let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra over a field of characteristic $p > 2$ with enveloping algebra $u(L)$. In our next result we characterize L for which $u(L)$ satisfies the Engel condition. Our results complement the results of [15,16] where it is determined when $u(L)$ satisfies a non-matrix identity or when $u(L)$ is Lie solvable, Lie nilpotent, or Lie super-nilpotent. Similar results for group rings and enveloping algebras of restricted Lie algebras were carried out in [3,6] and [8], respectively.

Theorem 2. *Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra over a field of characteristic $p > 2$. The following conditions are equivalent:*

1. *$u(L)$ is bounded Lie Engel;*
2. *$u(L)$ is PI, (L_0, L_0) is p-nilpotent, there exists an integer n such that $(\mathrm{ad}x)^n = 0$ for every $x \in L_0$, and either (L_1, L_1) is p-nilpotent or $\dim L_1 \leq 1$ and $(L_1, L_0) = 0$;*
3. *$u(L)$ is PI, L is nilpotent, (L_0, L_0) is p-nilpotent, and either (L_1, L_1) is p-nilpotent or $\dim L_1 \leq 1$ and $(L_1, L_0) = 0$.*

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