On regular subgroups of the affine group

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Let V be a vector space over a field F. Clearly, the group $T(V)$ of all translations of V and any of its coniugate subgroups by an element of $GL(V)$ are abelian regular subgroups of the affine group $AGL(V)$. Caranti, Dalla Volta and Sala [3] obtained a simple description of the abelian regular subgroups of the affine group $AGL(V)$ in terms of commutative associative radical algebras with underlying vector space V. As an application, they provide an example of an abelian regular subgroup of the affine group over an infinite vector space that trivially intersects the group of translations. It is well-known that such an example cannot be given when the vector space is allowed to be finite dimensional. To answer a question which appeared in [11], Hegedűs [8] constructed regular subgroups of some affine groups over a finite vector space containing only the trivial translation. Of course, his construction leads to non-abelian examples. Other examples can be found, for instance, in [7] and [17]. A description of the regular subgroups, not necessarily abelian, of the affine group is obtained by the first author and Rizzo [4] in terms of radical circle algebras, a generalization of radical algebras. A vector space V over a field F with a multiplication $\cdot$ is called a *(right) circle algebra* if the following statements hold:

(1) $\alpha(u \cdot v) = (\alpha u) \cdot v$,

(2) $(u + v) \cdot w = u \cdot w + v \cdot w$,

(3) $u \cdot (v + w + v \cdot w) = u \cdot v + u \cdot w + (u \cdot v) \cdot w$,

for all $\alpha \in F$ and for all $u, v, w \in V$. It is clear that any associative algebra is a circle algebra and that any commutative circle algebra is an associative algebra. This new structures are very closely related to (right) braces, introduced by Rump [14] to find non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation. So, using Rump's terminology [16], (right) circle algebras hereafter are called *(right) braces over* F or F*-braces*. Like in an ordinary algebra, let us introduce the *circle operation* in an F-brace V defined by $u \circ v := u + v + u \cdot v$, for all $u, v \in V$. Then $(V, \circ)$ is a semigroup. In particular, if $(V, \circ)$ is a group, then we say that the F-brace V is *radical*. The main result of [4] establishes the following link between regular subgroups of the affine group $AGL(V)$ and F- brace structures with the underlying vector space V.

Theorem[4] Let V be a vector space over a field F. Denote by $\mathcal{RB}$ the class of radical F-braces with underlying vector space V and by $\mathcal{T}$ the set of all regular subgroups of the affine group $AGL(V)$.

(a) If $V^\bullet$ is a radical F-braces with underlying vector space V, then $T(V^\bullet) = \{\tau_x | x \in V\}$, where $\tau_x : V \to V, y \mapsto y \circ x$, is a regular subgroup of the affine group $AGL(V)$.

(b) The map

$$f : \mathcal{RB} \longrightarrow \mathcal{T}, \quad V^\bullet \longmapsto T(V^\bullet)$$

is a bijection. In this correspondence, isomorphism classes of F-braces correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.

We note that, recently, a generalization of the above result to the holomorph of abelian group has been obtained in [12].

Then, the open problem of determining all regular subgroups of an affine group formulated in [10], may be replaced by that of determining all radical F-braces. Thus, the new constructions of radical F-braces presented in this paper allows to obtain rather systematic constructions of regular subgroups of the affine group. In particular, this approach allows to put in a more general context the regular subgroups constructed in [17].

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