Construction of quasi–linear left cycle sets

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In order to find new solutions of the Yang-Baxter equation, Drinfeld [4] asked the question of finding the so-called left theoretic solutions on an arbitrary non-empty set. Let X be a non-empty set and $\lambda_x : X \to X$, $\rho_x : X \to X$ and $r : X \times X \to X \times X$ maps such that $r(x, y) = (\lambda_x(y), \rho_y(x))$. Then (X, r) is called a *left non-degenerate involutive set-theoretic solution* of the Yang-Baxter equation if

1) $\forall x \in X \quad \lambda_x \in Sym_X$;
2) $r^2 = id_{X\times X}$;
3) $r_1r_2r_1 = r_2r_1r_2$, where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.

There are many papers on this subject and with many links to different topics. We mention here that Rump [8] showed that there is a bijective correspondence between left non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation and left cycle sets. Recall that a set X with a binary operation $\cdot$ is a *left cycle set* if the left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x \cdot y$ is invertible and the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

holds for all $x, y, z \in X$. Then the map $r : X \times X \longmapsto X \times X$, defined by $r(x, y) = (\lambda_x(y), \rho_y(x))$, where $\lambda_x(y) := \sigma_x^{-1}(y)$ and $\rho_y(x) := \lambda_x(y) \cdot x$, is a left nondegenerate involutive set-theoretic solution of the Yang-Baxter equation. Conversely, let (X, r) be a left non-degenerate involutive set-theoretic solution and put $x \cdot y := \lambda_x^{-1}(y)$, for all $x, y \in X$. Then X with the binary operation $\cdot$ is a left cycle set.

In this context, Rump introduced the *linear left cycle sets*, i.e. left cycle sets $(A, \cdot)$, where A is an additive abelian group such that the equations

$$a\cdot(b+c) = a\cdot b+a\cdot c \quad \text{and} \quad (a+b)\cdot c = (a\cdot b)\cdot(a\cdot c).$$

hold for all $a, b, c \in A$. Results on linear left cycle sets have appeared recently in several papers [1–3,7,10–14].

In [15], Rump introduced a new class of left cycle sets with an underlying abelian group structure. An additive abelian group A with a multiplication $\cdot$ is called a *quasi-linear left cycle set* if $(A, \cdot)$ is a left cycle set and the equation

$$a \cdot (b + c) = a \cdot b + (a - b) \cdot c.$$

holds for all $a, b, c \in A$. Any automorphism τ of an abelian group A makes A into a quasi-linear left cycle set with $a \cdot b = \tau(b)$, for all $a, b \in A$. We will refer to this as a *basic* quasi-linear left cycle set.

For an arbitrary quasi-linear left cycle set A, let us consider two ideals of A, the *socle* $Soc(A) := \{a | a \in A,\ \forall b \in A\ a \cdot b = 0 \cdot b\}$ and the *radical* $Rad(A)$, which is the additive subgroup generated by the differences $0 \cdot a - a$ with $a \in A$. Moreover, the *fixator* $Fix(A)$ is the set consisting of all elements $a \in A$ such that $b \cdot a = a$ for all $b \in A$. Then $Fix(A)$ is contained in $Soc(A)$ and it is an ideal provided that A is finite.

In this paper we produce a method to construct quasi-linear left cycle sets A with $Rad(A) \subseteq Fix(A)$. Moreover, among these cycle sets, we give a complete description of those for which $Fix(A) = Soc(A)$ and the underlying additive group is cyclic. Using such left cycle sets, we obtain left non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation which are different from those obtained in [5] and [6].

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