On filtered multiplicative basis of some associative algebras

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Let A be an associative algebra over a field F and denote by $\mathrm{Rad}(A)$ the Jacobson radical of A. An F-basis $\mathfrak{B}$ of A is called *multiplicative* if $\mathfrak{B}\cup\{0\}$ is a semigroup under the product of A. If one also has that $\mathfrak{B}\cap\mathrm{Rad}(A)$ is an F-basis of $\mathrm{Rad}(A)$, then $\mathfrak{B}$ is said to be a *filtered multiplicative basis* (shortly, f.m.b.) of A. Filtered multiplicative bases arise in the representation theory of associative algebras and were introduced by H. Kupisch in [8].

In [3], R. Bautista, P. Gabriel, A. Roiter and L. Salmeron proved that if a finite-dimensional associative algebra A has finite representation type over an algebraically closed field F, then A has an f.m.b. This implies that the number of isomorphism classes of algebras of finite representation type of a given dimension is finite and reduces the classification of these algebras to a combinatorial problem. In the same paper [3] it was asked when a group algebra has an f.m.b. and such a problem (not necessarily for group algebras) has been subsequently considered by several authors: see e.g. [1,2,4,5,7,9,11,12]. In particular, it is still an open problem whether a group algebra FG has an f.m.b. in the case when F is a field of odd characteristic p and G is a nonabelian p-group (see [10], Question 5). Moreover, in [6] the same problem was investigated in the setting of restricted enveloping algebras $u(L)$, where L is in the class $\mathfrak{F}_p$ of finite-dimensional and p-nilpotent restricted Lie algebras over a field of positive characteristic p. In particular, we characterized commutative restricted enveloping algebra having an f.m.b., and showed that if L has nilpotency class 2 and $p > 2$ then $u(L)$ does not have any f.m.b.

The aim of the present paper is to provide some further contribution on the problem of existence of an f.m.b. for an associative algebra. First, we deal with the conditions under which the property of having a multiplicative basis is inherited by homomorphic images. This result is then used to establish when a restricted enveloping algebra $u(L)$ has an f.m.b., where $L \in \mathfrak{F}_2$ has nilpotency class 2 over a field of characteristic 2, thereby complementing the previous results in [6]. Next, we show that if a finite-dimensional associative algebra A admits an f.m.b., then so does its graded algebra associated to the filtration given by the powers of the Jacobson radical. The combination of such a result with [6] allows to conclude that if F is a field of odd characteristic p and G is a finite p-group of nilpotency class 2, then the group algebra FG has no f.m.b., which provides a partial answer to the question 5 in [10].

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