The Integrable sector of the Skyrme - Faddeev model and Symmetry Invariant Discretization

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1. Introduction

The research activity in which the author was involved in the last year concern two main topics: the study of the integrable sector of the Skyrme - Faddeev model and the determination of Symmetry Invariant Discretization Schemes. Although apparently well separated, the two topics are interrelated by the fact that, outside of their possible integrable reductions, highly complex models need to be treated numerically . But since this is usually resources consuming and alternative algorithms and methods may significantly improved, by exploiting local point and more general symmetries in a systematic way.

In the present contribution Section 2 is is based on the results of the paper [1], where the 4-dimensional relativistic Skyrme-Faddeev model was studied with the aim to find its integrable reductions. In Section 3 a new symmetry invariant discretization method is presented, on the base of the results in the article [2]. As theoretical laboratory for application of the method, the Liouville equation was considered and different integration schemes were compared.

2. Integrable sector of the Skyrme-Faddeev model

The Skyrme-Faddeev model [3] is defined by the action

$$\mathcal{L} = \frac{1}{32\pi^2}\left(\partial_\mu\phi\cdot\partial^\mu\phi - \kappa\,(1-\phi\cdot\phi) - \frac{\lambda}{4}\,(\partial_\mu\phi\times\partial_\nu\phi)\cdot(\partial^\mu\phi\times\partial^\nu\phi)\right),$$

where $\lambda > 0$ is a scaling parameter, determining the breaking of the conformal symmetry. Then its finite energy localized excitations have typical size $\approx \sqrt{\lambda}$ (*skyrmion*). The Lagrangian multiplier κ implements the unimodular condition $\phi\cdot\phi = 1$, than the polar representation $\phi = (\sin w\cos u, \sin w\sin u, \cos w)$, for w and u suitable functions, is introduced. The Euler–Lagrange equations read

$$\begin{gathered}\partial_\mu w^\mu = \tfrac{1}{2}\sin(2w)u_\nu u^\nu + \\ \tfrac{\lambda}{2}\sin w\; u_\nu\,\partial_\mu[\sin w(w^\mu u^\nu - w^\nu u^\mu)], \\ w_\mu u^\mu\sin(2w) + \sin^2 w\;[\partial_\mu u^\mu + \\ \tfrac{\lambda}{2}w_\nu\partial_\mu(u^\mu w^\nu - u^\nu w^\mu)] = 0 \qquad .\end{gathered}$$

These equations are not integrable in general. In particular the skyrmion solutions indexed by the Hopf topological charge (the hopfions) are numerically computed [4,5]. We found good approximated expressions of them [6].

Actually, dimensional reductions of the equations will lead to extended infinite energy solutions. However, we are motivated by: i) characterize special completely integrables subsectors and find analytical solutions, ii) give a suitable mathematical description of the of the extended phases observed in ferromagnets and multiferroincs [**?**, **?**], iii) describe the observed interaction among hopfions, waves and/or magnetic domains.

2.1. Domain-Walls

Limiting ourselves to the most important case, let us vanish the coefficients of all functions of w in (2.1), leading to the overdetermined quasilinear system

$$\begin{aligned}&\partial_\mu w^\mu = 0, && w_\mu w^\mu = -\epsilon^2, \\ &u_\mu w^\mu = 0, && a_\mu w^\mu = 0 \quad \text{with} \quad a = u^\nu u_\nu,\end{aligned}$$

where $\epsilon^2 = \frac{2}{\lambda}$ for notational clarity. The first two equations form the d'Alembert-Eikonal system [7], the general solution of which is given in the implicit form via the auxiliary τ by

$$\begin{gathered}w = x^k A_k(\tau) + A_0(\tau), \\ x^0 = x^k B_k(\tau) + B_0(\tau), \\ (A_i) = \sqrt{\tfrac{2}{\lambda}}\;\hat{\mathbf{A}}\,(f\,(\tau)\,, g\,(\tau)))\;\left(\hat{\mathbf{A}}^2 = 1\right), \\ (B_i) = \hat{\mathbf{B}} = \pm\tfrac{\hat{\mathbf{A}}\times\hat{\mathbf{A}}'}{|\hat{\mathbf{A}}\times\hat{\mathbf{A}}'|}\,[f\,(\tau)\,, g\,(\tau))]\,,\end{gathered}$$

where $f\,(\tau)$, $g\,(\tau)$, $A_0\,(\tau)$ and $B_0\,(\tau)$ are arbitrary differentiable functions. An analysis on the solutions has been partially performed. In particular, if $B_0(\tau)$ is not an injective continuous function, caustics and singularities of the wave front may appear. We know that in general the singularities of the wave fronts are classified by the Coxeter groups [8].

Concerning the residual subsystem, cross differentiations of the above equations and a systematic substitutions of the x^0-derivatives lead to a set of compatibility conditions, which is the Monge-Ampére equation $\mathrm{Det}\,[w_{ij}] = 0$, and the involved quadratic constraints for the derivatives u_k. They can be simplified and possibly solved, if one can find a first order linear system of the form

$$u_0 = Au_1, \quad u_2 = Bu_1, \quad u_3 = Cu_1,$$

where the functions $A,\ B,\ C$ depend w_m and $\partial_n w_m$ only. This strategy leads to the general solution for u in 2-dimensional space, i.e.

$$u = F\,[w_1, w_2]\,,\ F \text{ arbitrary real differentiable}$$

In 3 dimensions, a similar analysis is much more complicated, but also in that case the u is completely determined as an arbitrary function of two derivatives of w only.

2.2. Waves

Looking for invariant solutions of any 2-dimensional sub-algebra of the translational symmetries of the Skyrme-Faddeev model, one is lead to invariant reduction

$$w = \Theta\,[\theta]\,,\ u = \Phi\,[\theta]+\tilde{\theta}, \qquad \theta = \alpha_\mu x^\mu, \tilde{\theta} = \beta_\mu x^\mu$$

in which one distinguishes θ as the *phase* from the *pseudo-phase* $\tilde{\theta}$. The corresponding 3-parametric ODE system arises

$$\left[2B_3 - \tfrac{\lambda}{4}\mathcal{B}\sin^2\Theta\right]\Theta_{\theta\theta} =$$
$$\sin 2\Theta\left(\tfrac{\lambda}{8}\mathcal{B}\,\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right)$$
$$2B_3\sin^2\Theta\;\Phi_{\theta\theta} + \Theta_\theta\sin 2\Theta\,(2B_3\Phi_\theta + B_2) = 0,$$

where $B_1 = -\beta_\mu\beta^\mu$, $B_2 = -2\alpha_\mu\beta^\mu$ $B_3 = -\alpha_\mu\alpha^\mu$ and $\mathcal{B} = B_2^2 - 4B_1B_3$. These system posses 4 integrals of motion and can be simplified to the form

$$\Theta = \arcsin\sqrt{\psi},\ \Phi = -\tfrac{B_2U_2}{2B_3}\left[\int\tfrac{d\theta}{\psi(\theta)} + \theta\right]$$
$$\psi_\theta^2 = \tfrac{64(\psi-1)(\psi-A_1)(\psi-A_2)}{\lambda^2\mathcal{B}\psi_1(\psi_1-\psi)}, \qquad (0 \le \psi \le 1)$$

where the constants $U_{1,2}$, $A_{1,2}$ and $\psi_1 = \frac{8B_3}{\lambda\mathcal{B}}$ are related to the integrals of motion. Those equations are integrated in terms of incomplete elliptic integrals of the third kind, and the general solution leads to three different linear harmonic branches in the limit of ψ_1 is approaching A_1, A_2 and 1 respectively. However, for $\psi_1 \to 1$ the exact simplest solution is derived

$$\psi = \frac{1}{2}\left((A_1 - A_2)\cos\left(\frac{8}{\sqrt{\mathcal{B}}\lambda}\theta\right) + A_1 + A_2\right),$$

leading to spin waves with two independent planar modes (sometimes called *cyclonic* and *extra-cyclonic* [9]) At the opposite, one can notice from the general solution admits infinite wavelength limits, expressed in terms of elementary hyperbolic functions, similar to localized solitons. Thus, one can conjecture that varying in a suitable way all parameters, classes of slowly deformed periodic solutions can be found. To do this, we obtained a set of ODE's for the slowly changing parameters, by resorting to the Whitham average method [1].

3. Invariant Symmetry Discretization

As said in the Introduction, there is a general program on the study of continuous symmetries of discrete equations and on the symmetry preserving discretization of differential equations [10,11]. This program has several aspects including i) discretize field theories on a discrete space–time preserving continuous symmetries, ii) improve numerical methods of solving specific ordinary and partial differential equations, by incorporating important qualitative features of these equations. Such features may be integrability, linearizability, Lagrangian or Hamiltonian formulation, or some other features. One possible way of doing this is to not use a preconceived constant lattice, but construct an invariant set of equations defining both the lattice and system of difference equations. The lattice thus appears as part of a solution of a set of discrete equations and the symmetry group acts on the solutions of the equation and on the lattice. We concentrate on the preservation of Lie point symmetries. In our case the idea is to take an ordinary or partial differential equation (ODE or PDE) with a known Lie point symmetry algebra $\mathcal{L}$ realized by vector-fields. The differential equation is then approximated by a difference system with the same symmetry algebra. Then, a difference system is constructed out of the invariants of the Lie point symmetry group $\mathcal{G}$ of the original ODE (PDE). The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is realized by the same vector fields as for the continuous equation, however its action is prolonged to all points of the lattice, rather than to derivatives. To explore such a procedure, we considered the completely integrable hyperbolic Liouville equation in algebraic version

$$u\,u_{xy} - u_x\,u_y = u^3, \tag{3.1}$$

since we can easily compare analytic solutions with those on a lattice and because one may describe in detail the breaking of symmetry in a discretization procedure. In fact, its general solution is $u = 2\frac{\phi_{1,x}\,\phi_{2,y}}{(\phi_1+\phi_2)^2}$, for arbitrary differentiable functions $\phi_1(x)$ and $\phi_2(y)$, and its point symmetry algebra is isomorphic to the direct sum of two Virasoro algebras $L = vir_x \oplus vir_y$ and its maximal finite dimensional subalgebra is $sl_x(2,\mathbb{R}) \bigoplus sl_y(2,\mathbb{R})$. The basic idea of the invariant discretization of a PDE is to replace it by a system of difference equations, formed out of invariants of the action of the symmetry group of the PDE. This difference system (ΔS) describes both the original PDE and a lattice. In the case of a 2nd-order PDE in 2 variables like (3.1), the ΔS will have the form

$$E_\alpha(x_{m+i,n+j}, y_{m+i,n+j}, u_{m+i,n+j}) = 0,$$
$$\alpha = 1,\ldots,N,\ i_{min} \le i \le i_{max},\ j_{min} \le j \le j_{max}.$$

This difference system is written on a *stencil*: a finite number N of adjacent points, sufficient to reproduce, in the continuous limit, all derivatives figuring in the differential equation. In order to obtain an *invariant* ΔS we must construct it out of difference invariants of the Lie point symmetry group $\mathcal{G}$ of the PDE. To calculate these invariants we consider the action of the symmetry vector fields $\hat{Z}^a$ at some reference point $\{x_{0,0},\ y_{0,0},\ u_{0,0}\}$ and prolong them to all points in a chosen stencil, amounting to a prolongation to the discrete jet space $\mathrm{pr}\hat{Z}^a = \sum_{i,j}(\xi^a_{i,j}\partial_{x_{i,j}} + \eta^a_{i,j}\partial_{y_{i,j}} + \phi^a_{i,j}\partial_{u_{i,j}})$. The invariants on the stencil are obtained by solving the equations $\mathrm{pr}\hat{Z}_a I(x_{i,j}, y_{i,j}, u_{i,j}) = 0$. Weak invariant, i.e. invariants for very special values are also admissible. Working on a 4-point stencil, and restricting to $sl_x(2,\mathbb{R}) \bigoplus sl_y(2,\mathbb{R})$ subalgebra, the corresponding group acts transitively on the space of the continuous variables $(x, y, u) \in \mathbb{R}^3$, and sweeps out an orbit of codimension 6 on the 12-dimensional space of all the variables on the 4-point stencil. Hence we obtain 6 functionally independent invariants. Two of them can be used to define an invariant lattice. However, They are in fact only weak invariants under the Virasoro algebra. This constraints fixes the lattice to be orthogonal. The remaining invariants $sl_x(2,\mathbb{R}) \bigoplus sl_y(2,\mathbb{R})$ are not longer Virasoro invariants. However, certain functions of them approximate the Liouville equation to order $O\left(h^3k^3\right)$ in the lattice meshes h and k. So, one is lead to a $SL_x(2) \otimes SL_y(2)$ invariant difference scheme, not however Virasoro invariant. The scheme is suitable for solving various types of boundary value problems, giving u_{ik} on the axes, or on upward or downward staircases. One of the obtained recursion formula is

$$u_{1,1} = \frac{u_{0,1}u_{1,0}\left(ah_{1,0}k_{0,1}\sqrt{|u_{0,1}u_{1,0}|}+1\right)}{u_{0,0}\left((a-1)h_{1,0}k_{0,1}\sqrt{|u_{0,1}u_{1,0}|}+1\right)}.$$

We checked such a formula over a certain set of test functions and we show that, under some supplementary conditions the invariant scheme provides a much better approximation of exact solutions than a comparable standard (non invariant) scheme and also than a scheme invariant under an infinite dimensional group of generalized symmetries introduced by Rebelo and Valiquette.

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