On the Stackelberg Fuel Pricing Problem

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A fundamental decisional process in many business activities concerns how to set the selling prices so as to maximize its own revenue, once knowing the selling prices of the competitors and the customers' preferences. The scenario in which the latter are implicitly defined in terms of some optimization problem are usually referred to as *Stackelberg pricing problems* [6]. These problems can be modeled as multi-player one-round games in which there is a special player, the *leader*, while all the others are *followers*. The first action is undertaken by the leader who decides on some parameters (e.g., the selling prices) and then the followers respond by deciding their actions. Each follower adopts, as her action, the optimal solution of a certain optimization problem (e.g., satisfying her own demand at the minimum cost) which depends on some of the parameters fixed by the leader and on some other values on which the leader has no control (e.g., the selling prices of the competitors). Since the followers' choices influence, in turn, the leader's revenue, the determination of the best possible action for the leader often results in a challenging algorithmic problem.

A considerable research attention has been devoted in the last years to the study of Stackelberg network pricing problems based on some fundamental (polynomial time solvable) optimization problems, such as shortest paths, shortest path trees and minimum spanning trees. In this paper, we study the *Stackelberg fuel pricing problem* (SFPP) which is a Stackelberg network pricing problem based on the *gas station problem* (GSP): an optimization problem introduced in [5] to model situations in which drivers have to go from one location to another and have to decide where to fill their cars with fuel so as to minimize the travel cost, once knowing the fuel prices at the various gas stations along the road network.

Model and Notation. For a positive integer k, let $[k]$ denote the set $\{1, \dots, k\}$. A *road network* is an edge-weighted directed graph $G = (V, E, w)$, with $|V| = n$, $|E| = m$ and $w : E \to \mathbb{R}_{\geq 0}$ such that, for each $e = (u, v) \in E$, $w(e)$ specifies the amount of fuel (expressed in gallons) needed to go from u to v.

An instance (G, s, t, S, p) of the GSP is defined by a road network G, a pair of nodes $s, t \in V$, a set of nodes $S \subseteq V$ and a function $p : S \to \mathbb{R}_{\geq 0}$. The set of nodes S represents the locations of gas stations and the function p models the selling prices (per gallon) at each of the gas stations. There is a driver who needs to go from s to t. For the sake of simplicity, we assume that the driver's car is equipped with a tank of unlimited capacity and that $s \in S$, so that the driver can fill with as much fuel as she wants at price $p(s)$ when starting her trip. The driver wants to determine the best possible *itinerary*, that is, which (s, t)-path to drive through and which gas stations to stop at for fueling so as to minimize the total fuel cost[1].

An instance $(G, (s_i, t_i, \lambda_i)_{i \in [k]}, L, C, p_C, e)$ of the SFPP is defined by a road network G, k source-destination pairs (s_i, t_i) with an associated integer weight $\lambda_i \geq 1$ for each $i \in [k]$, two sets $L, C \subseteq V$, a function $p_C : C \to \mathbb{R}_{\geq 0}$ and a value $e \geq 0$. The sets of nodes L and C represent the locations of gas stations: the gas stations located at nodes in L are owned by the leader, while those located at nodes in C are owned by her competitors, so that the function p_C models the selling prices established by the competitors at each of their gas stations[2]. Note that we do not require L and C to be disjoint, i.e., either the leader and one of her competitors may own a gas station at the same location. The leader buys (or produces) the fuel at price e per gallon. There are k types of drivers (the followers) such that, for each $i \in [k]$, the driver of type i wants to go from s_i to t_i along the cheapest path in G, where λ_i denotes the number of drivers of type i, that is, how many drivers want to go from s_i to t_i. Once the leader has established a pricing function $p_L : L \to \mathbb{R}_{\geq 0}$ defining the fuel prices at her own gas stations, each follower of type $i \in [k]$ determines her action by solving the instance $(G, s_i, t_i, L \cup C, p)$ of the GSP, in which, for each $v \in L \cup C$, $p(v) := \min\{p_L(v), p_C(v)\}$. The leader has to determine the pricing function

[1]The amount of fuel bought at each gas station s is implicitly defined by the minimum distance between s and the successive station chosen for fueling

[2]In the case in which more than one competitor owns a gas station in a given location v, $p_C(v)$ will denote the cheapest fuel price among them.

$p_L : L \to \mathbb{R}_{\geq 0}$ providing her with the highest possible revenue. To this aim, we make the following simplifying assumption which is common in the setting of Stackelberg pricing problems: when the gas station problem has more than one optimal solution, the follower always chooses the one providing the leader with the highest revenue[3]. Furthermore, we assume that $s_i \in C$ $\forall i \in [k]$, otherwise, either the problem is not feasible or the revenue is unbounded.

More formally, given a node $v \in V$ and a pricing function p_L for the leader, let $q(p_L, v) := 1_{p_L(v) \leq p_C(v)}$. For a pair of nodes $u, v \in V$, let $d(u, v)$ be the distance from u to v in G and $\mathcal{B}(u, v)$ be the set of itineraries connecting u to v, that is, the set of sequences of nodes $[u = v_{i_1}, v_{i_2}, \ldots, v_{i_r} = v]$ such that $v_{i_s} \in L \cup C$ for each $s \in [r-1]$. Set $p_s := p(v_{i_s})$, $d_s := d(v_{i_s}, v_{i_{s+1}})$ and $q_s := q(p_L, v_{i_s})$ for each $s \in [r-1]$. Given p_L and $B \in \mathcal{B}(u, v)$, a driver choosing itinerary B experiences a cost $c(p_L, B)$ and yields a contribution $g(p_L, B)$ to the leader's revenue which are defined as follows:

$$c(p_L, B) := \sum_{s=1}^{r-1} p_s \cdot d_s$$

and

$$g(p_L, B) := \sum_{s=1}^{r-1} (p_s - e) \cdot d_s \cdot q_s.$$

Let $cg(p_L, B) = (c(p_L, B), g(p_L, B))$ and define a total ordering relation $\prec$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that $[x_1, y_1] \prec [x_2, y_2] \Leftrightarrow x_1 < x_2 \vee \{x_1 = x_2 \wedge y_1 > y_2\}$. Let $\mathcal{B}^*(u, v) \subseteq \mathcal{B}(u, v)$ be the set of itineraries $B \subseteq \mathcal{B}(u, v)$ minimizing $cg(p_L, B)$ according to the ordering relation $\prec$. Let $cg(p_L, u, v) := [c(p_L, u, v), g(p_L, u, v)] := \min_{B \in \mathcal{B}(u,v)} cg(p_L, B)$. Given a driver of type i, we have that $c(p_L, s_i, t_i)$ is the cost of her cheapest path, and $g(p_L, s_i, t_i)$ is her contribution to the leader's revenue, so that $g(p_L) := \sum_{i=1}^{k} \lambda_i \cdot g(p_L, s_i, t_i)$ is the leader's total revenue that has to be maximized. Let $p_M := \max_{i \in [k]} p_C(s_i)$. It is easy to prove that, if we set $p_L(v) > \min\{p_M, p_C(v)\}$ for some $v \in L$, no driver will stop at station v for fueling. Similarly, whenever $p_C(v) > p_M$ for some $v \in C$, no driver will stop at station v for fueling. Therefore, we can suppose without loss of generality that $p_C(v) \leq p_M$ $\forall v \in C$, the leader's revenue $g(p_L)$ is bounded and, in order to be maximized, p_L can be chosen in such a way that $e \leq p_L(v) \leq p_C(v)$ $\forall v \in L$.

Related Work. Briest, Hoefer and Krysta author an influential work [3] on Stackelberg network pricing problems which widely generalizes previous results in the field. In particular, they consider the case in which the edges of the network are partitioned into two sets: the set of *fixed-price edges* and that of *priceable edges*, with the latter owned by the leader. Each follower buys a subnetwork of minimum cost and so the leader wants to assign suitable prices to the priceable edges so as to maximize her revenue. They study the approximation guarantee of the *single-price algorithm* in this general class of problems. This algorithm, which assigns the same (suitably computed) price to all priceable edges, has been first analyzed in [4] for the case of a single follower buying a minimum spanning tree. Briest, Hoefer and Krysta show that, for the case of a single follower, the approximation guarantee is $(1+\epsilon)H_h$, where $\epsilon > 0$ is an arbitrary value, h is the number of priceable edges and H_i is the ith harmonic number, while, for the case of k followers, it becomes $(1+\epsilon)(H_h + H_k)$. Finally, when the followers may have different weights, they show that the single-price algorithm achieves an approximation guarantee of $(1+\epsilon)h^2$ and also provide a lower bound of $O(h^\epsilon)$ on the approximability of the problem.

Determining whether there are approximation algorithms better than the single-price one is, perhaps, the most important open problem in this field of research. In fact, while the performance of this algorithm remains essentially the same even when instantiated to specific optimization problems such as shortest paths, shortest path trees and minimum spanning trees, the impossibility results known so far in these cases only refer to APX-hardness, see [1,2,4].

Our Contribution. We show that the SFPP is APX-hard even in the basic case in which the road network is modeled by an undirected planar graph and the competitors discriminate on two different selling prices only, by means of a reduction from the maximum independent set problem on cubic graphs. This reduction, however, requires that $|L|$ is a non-constant value. This assumption is essential, anyway, since, for the case in which $|L| = O(1)$, we show that the problem can be solved in polynomial time.

We stress that the SFPP does not fall within the scope of the Stackelberg network pricing problems defined by Briest, Hoefer and Krysta in [3] and that the presence of additional parameters in the definition of the problem (in particular, the edge-weights) makes the performance of the single-price algorithm unlikely to be uninfluenced by the characteristics of the road network given in input. To this aim, we define a general class of Stackelberg network pricing problems which extends the one given by Briest, Hoefer and Krysta and includes the SFPP. For this class of problems,

[3] In fact, if this is not the case, the leader can decrease some selling price of a negligible amount so as to achieve almost the same revenue as in the case in which the assumption holds.

we show that the single-price algorithm provides an approximation guarantee which is logarithmic in some parameters of the input instance and that this bound is tight.

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