Second order elliptic operators with diffusion coefficients growing as $|x|^\alpha$ at infinity

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In [1], we studied elliptic operators with unbounded diffusion coefficients of the form

$$Lu = (1+|x|^2)^{\frac{\alpha}{2}} \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} u, \qquad (1)$$

for values of $\alpha > 2$ and $A = (a_{ij}) : \mathbb{R}^N \to \mathbb{R}^{N^2}$ on $L^p(\mathbb{R}^N) = L^p(\mathbb{R}^N, dx)$ and with respect to the Lebesgue measure. We assumed that (a_{ij}) is symmetric and satisfies the uniform ellipticity condition. Moreover we required that the coefficients a_{ij} admit a limit a^0_{ij} when x goes to infinity. We have been interested both in parabolic problems associated with L

$$u_t - Lu = 0 \qquad u(0) = f$$

and in the solvability of the elliptic equation

$$\lambda u - Lu = f \qquad \lambda \in \mathbb{C}.$$

These problems are respectively connected to the generation of analytic semigroups and to the spectral property of L. In particular, we considered generation of analytic semigroups via the spectral property of L.

The case $a_{ij} = \delta_{ij}$ (Kronecker's delta), that is, the operator $(1+|x|^\alpha)\Delta$, has been widely investigated. Among other properties, generation results of analytic semigroups in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ when $\alpha \le 2$ and for $p > \frac{N}{N-2}$ when $\alpha > 2$ have been proved. It is important to notice that the restriction on p in the last case is sharp. A more general theory concerning operators containing, in addition, a drift term and allowing a degeneration near the origin has also been developed.

The replacement of the Laplacian with an elliptic operator of the second order does not seem to be immediate, even for $\alpha \le 2$. Indeed the techniques for proving the previous results are based on homogeneous Calderón-Zygmund inequalities that do not hold for more general pure second order elliptic operators. By using homogeneous second order derivatives estimates for purely second order operators whose coefficients admit limit at infinity, the case $\alpha \le 2$ has already been treated in a previous paper where the author characterized the domain of the operator via weighted a-priori estimates and proved existence and generation results by modifying classical methods for elliptic and parabolic linear problems. A completely different approach has been needed when $\alpha > 2$. Moreover, the methods applied in that case for the operator $(1+|x|^\alpha)\Delta$ are strongly related with the explicit representation of the fundamental solution of the Laplacian and cannot be easily extended to the general case $a_{ij} \ne \delta_{ij}$.

We proved that, for $\frac{N}{N-2} < p < \infty$, the operator L generates an analytic semigroup in $L^p(\mathbb{R}^N)$. Moreover, according with the previous remarks, we provided an explicit description of the domain when $\alpha < \frac{N}{p'}$.

REFERENCES

1. M. Sobajima, C. Spina: Second order elliptic operators with diffusion coefficients growing as $|x|^\alpha$ at infinity, Forum Mathematicum, to appear.