Complexity of Rational and Irrational Nash Equilibria

Vittorio Bilò [1] and Marios Mavronicolas[2]

[1]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Italy

[2]Department of Computer Science, University of Cyprus, Nicosia, Cyprus

Understanding the complexity of algorithmic problems pertinent to equilibria in (finite) strategic games is one of the most intensively studied topics in *Algorithmic Game Theory* today. Much of this research has focused on *Nash equilibria*, the most influential equilibrium concept in *Game Theory*. In the wake of the complexity results for *search problems* about Nash equilibria, a series of breakthrough works [1,4] shows that, even for *two-player* games, computing an (exact) Nash equilibrium is complete for $\mathcal{PPAD}$, a complexity class to capture the computation of discrete fixed points; so also is computing an *approximate* Nash equilibrium.

In this work, we continue the study of the complexity of *decision problems* about Nash equilibria. The celebrated result of John Nash [7] shows that every (finite) game admits a *mixed* Nash equilibrium; so, it trivializes the decision problem about the existence of (mixed) Nash equilibria, while it simultaneously leaves open the complexity of decision problems about the existence of Nash equilibria with certain properties.

Gilboa and Zemel [6] were the *first* to present complexity results (more specifically, $\mathcal{NP}$-hardness results) about mixed Nash equilibria (and *correlated equilibria*) for games represented in explicit form; they identified some $\mathcal{NP}$-hard decision problems about the existence of (mixed) Nash equilibria with certain properties for *two-player* strategic games. (For example, it is $\mathcal{NP}$-hard to decide if a game admits a Nash equilibrium where each player receives utility above some threshold.)

Much later, Conitzer and Sandholm [2,3] provided a very notable *unifying reduction*, henceforth abbreviated as CS-reduction, to show that all decision problems from [6] and many more are $\mathcal{NP}$-hard. The CS-reduction [2,3] yields a two-player game out of a CNF formula ϕ; it is then shown that the game has a Nash equilibrium with certain properties (in addition to some fixed *pure* Nash equilibrium) if and only if ϕ is satisfiable. Hence, deciding the properties is $\mathcal{NP}$-hard.

The CS-reduction uses literals, variables and clauses from the formula ϕ, together with a special strategy f, as the strategies of each player. The essence of the CS-reduction is that *(i)* both players choosing f results in a pure Nash equilibrium, and *(ii)* a player could otherwise improve (by switching to f) unless both players only randomize over literals. More important, a Nash equilibrium where both players only randomize over literals is possible (and has certain properties) if and only if ϕ is satisfiable.

In this paper, we shall extend the work from [2,3,6] to decision problems pertinent to *rationality* and *irrationality* properties of mixed Nash equilibria. Recall that a Nash equilibrium is *rational* if all involved probabilities are rational and otherwise it is *irrational*; so, a pure Nash equilibrium is rational but not viceversa. All *two-player* games have only rational Nash equilibria, while there are already known three-player games with no rational Nash equilibrium (cf. [8]).

We introduce two new natural decision problems, denoted as $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH, respectively; these problems ask whether or not there is a rational (resp., irrational) Nash equilibrium[1]. Hence, both problems $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH trivialize when restricted to two-player games but become non-trivial for games with at least three players. Since the CS-reduction [2,3] applies to two-player games, it will not be directly applicable to settling the complexity of $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH.

To establish the $\mathcal{NP}$-hardness of the problems $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH we shall use two new suitable decision problems that make no reference to rationality or irrationality properties of Nash equilibria. These problems will be NASH-EQUIVALENCE and NASH-REDUCTION, respectively, but yet witness $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH, respectively. Both problems receive as input a pair of strategic games SG and $\widehat{\text{SG}}$ with the same number of players $r \geq 2$; they inquire about some mutual properties of their Nash equilibria. We shall use two distinct CS-like reductions with each simultaneously showing that both problems NASH-EQUIVALENCE and $\exists$ RATIONAL NASH (resp., NASH-REDUCTION and $\exists$ IRRATIONAL NASH)

[1]We were inspired to study these problems by a corresponding question posed by E. Koutsoupias to M. Yannakakis during his Invited Talk at *SAGT 2009*.

are $\mathcal{NP}$-hard.

To the best of our knowledge, these complexity results for $\exists$ RATIONAL NASH and $\exists$ IRRATIONAL NASH are the *first* $\mathcal{NP}$-hardness results for a decision problem inquiring the existence of a combinatorial object involving rational (resp., irrational) numbers; no such $\mathcal{NP}$-hard problems are listed in [5].

The problem NASH-EQUIVALENCE asks whether the sets of Nash equilibria for the games SG and $\widehat{\mathsf{SG}}$ coincide. Fixing $\widehat{\mathsf{SG}}$ to some *gadget* game yields the restricted problem NASH-EQUIVALENCE($\widehat{\mathsf{SG}}$) with a single input SG.

Assume that *(i)* the set of Nash equilibria for $\widehat{\mathsf{SG}}$ is a subset of those for SG *and (ii)* $\widehat{\mathsf{SG}}$ has no rational Nash equilibrium, then SG may have a rational Nash equilibrium if and only if the set of Nash equilibria for SG and $\widehat{\mathsf{SG}}$ do *not* coincide. So, the existence of a rational Nash equilibrium for SG is a witness to the *non-equivalence* of SG and $\widehat{\mathsf{SG}}$ under the given assumptions. So, if NASH-EQUIVALENCE($\widehat{\mathsf{SG}}$) is $\mathcal{NP}$-hard, then so is $\exists$ RATIONAL NASH. We show that NASH-EQUIVALENCE($\widehat{\mathsf{SG}}$) is $\mathcal{NP}$-hard for an arbitrary but *fixed* strategic game $\widehat{\mathsf{SG}}$. Fixing $\widehat{\mathsf{SG}}$ to admit no rational Nash equilibrium yields that $\exists$ RATIONAL NASH is $\mathcal{NP}$-hard.

The problem NASH-REDUCTION asks whether there is a Nash reduction from SG to $\widehat{\mathsf{SG}}$. Roughly speaking, a ***Nash reduction*** consists of a family of surjective functions, one per player, mapping the strategy set of each player in SG to the strategy set of the same player in $\widehat{\mathsf{SG}}$. Note that any family of surjective functions induces a map from mixed profiles for SG to mixed profiles for $\widehat{\mathsf{SG}}$ in the natural way: probabilities to different strategies of a player in SG that map to the same strategy (of the player) in $\widehat{\mathsf{SG}}$ are added up. However, a Nash reduction must, in addition, **preserve** at least one Nash equilibrium: for any Nash equilibrium for $\widehat{\mathsf{SG}}$, there must be a Nash equilibrium for SG that maps to it.

Assume that *(i)* there is a Nash reduction from SG to $\widehat{\mathsf{SG}}$, *and (ii)* SG has only rational Nash equilibria. Then $\widehat{\mathsf{SG}}$ has at least one rational Nash equilibrium. It follows from the contraposition that, if $\widehat{\mathsf{SG}}$ has no rational Nash equilibrium, then either *(i')* there is no Nash reduction from SG to $\widehat{\mathsf{SG}}$ or *(ii')* SG has an irrational Nash equilibrium. Hence, the inexistence of an irrational Nash equilibrium for SG is a witness to the inexistence of a Nash reduction from SG to $\widehat{\mathsf{SG}}$. So, if NASH-REDUCTION($\widehat{\mathsf{SG}}$) is $\mathcal{NP}$-hard, then so is $\exists$ IRRATIONAL NASH.

We show that NASH-REDUCTION($\widehat{\mathsf{SG}}$) is $\mathcal{NP}$-hard for a *fixed* strategic game $\widehat{\mathsf{SG}}$ which *(a)* is constant-sum with sum $r \cdot u$, *(b)* has a unique Nash equilibrium which is *(b/i)* fully mixed and in which *(b/ii)* the utility of each player is u. Fixing the gadget $\widehat{\mathsf{SG}}$ so that, in addition, it admits no rational Nash equilibrium yields that $\exists$ IRRATIONAL NASH is $\mathcal{NP}$-hard.

Both reductions yield a game $\mathsf{SG} = \mathsf{SG}(\phi)$ with an *arbitrary* number of players $r \geq 2$ inherited from the gadget game $\widehat{\mathsf{SG}}$; recall that the CS-reduction yields a two-player game. Hence, the resulting game may or may not have properties, such as having a rational Nash equilibrium or having an irrational Nash equilibrium, which two-player games necessarily have or necessarily do not have, respectively.

For each player, the special strategy f from the CS-reduction [2,3] is replaced by the strategies of the player in the gadget game $\widehat{\mathsf{SG}}$. Two features of the reduction are: *(i)* If all players choose strategies as in a Nash equilibrium for $\widehat{\mathsf{SG}}$, the result is a Nash equilibrium for the resulting game $\mathsf{SG}(\phi)$. This implies that $\mathsf{SG}(\phi)$ necessarily has Nash equilibria with desirable properties (such as irrationality), as opposed to necessarily having a pure Nash equilibrium (in the case of the CS-reduction). *(ii)* It still holds that a player could otherwise improve (by switching to a strategy from the gadget game) unless all players only randomize over literals. More important, a rational Nash equilibrium where all player only randomize over literals is possible if and only if the formula ϕ is satisfiable.

For each player, the special strategy f from the CS-reduction [2,3] remains. A feature of the reduction is: *(i)* It still holds that a player could otherwise improve (by switching to the special strategy f) unless all players only randomize over literals.

We use a surjective map from (disjoint) sets of literals in ϕ to single strategies from the gadget game $\widehat{\mathsf{SG}}$. This allows using utilities from the gadget game to define utilities when all players choose literals. (In the CS-reduction, these utilities were identical for all players.)

So, if all players randomize over literals to induce a Nash equilibrium $\widehat{\boldsymbol{\sigma}}$ for $\widehat{\mathsf{SG}}$, the result is a profile $\boldsymbol{\sigma}$ for SG that preserves rationality (resp., irrationality) of $\widehat{\boldsymbol{\sigma}}$. More important, $\boldsymbol{\sigma}$ is an (irrational) Nash equilibrium for SG (and, hence, there is a reduction from SG to $\widehat{\mathsf{SG}}$) if and only if the formula ϕ is satisfiable.

REFERENCES

1. X. Chen, X. Deng and S.-H. Teng, "Settling the Complexity of Computing Two-Player Nash Equilibria," *Journal of the ACM,* Vol. 56, No. 3, 2009.
2. V. Conitzer and T. Sandholm, "Complexity Results about Nash Equilibria," *Proceedings*

of the 18th Joint Conference on Artificial Intelligence, pp. 765–771, August 2003.
3. V. Conitzer and T. Sandholm, "New Complexity Results about Nash Equilibria," *Games and Economic Behavior,* Vol. 63, No. 2, pp. 621–641, July 2008.
4. C. Daskalakis, P. W. Goldberg and C. H. Papadimitriou, "The Complexity of Computing a Nash Equilibrium," *SIAM Journal on Computing,* Vol. 39, No. 1, pp. 195–259, 2009.
5. M. R. Garey and D. S. Johnson, *Computers and Intractability — A Guide to the Theory of $\mathcal{NP}$-Completeness,* W. H. Freeman, 1979.
6. I. Gilboa and E. Zemel, "Nash and Correlated Equilibria: Some Complexity Considerations," *Games and Economic Behavior,* Vol. 1, No. 1, pp. 80–93, March 1989.
7. J. F. Nash, "Equilibrium Points in n-Person Games," *Proceedings of the National Academy of Sciences of the United States of America,* pp. 48–49, Vol. 36, 1950.
8. J. F. Nash, "Non-Cooperative Games," *Annals of Mathematics,* Vol. 54, No. 2, pp. 286–295, 1951.