Nonlinear Schrödinger equations and coupled Maxwell-Bloch systems

B. Prinari [1] and F. Vitale [2]

[1]Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento and Sezione INFN, Lecce - Italy
Department of Mathematics, University of Colorado at Colorado Springs - USA

[2]Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento and Sezione INFN, Lecce - Italy

1. Nonlinear Schrödinger systems

Nonlinear Schrödinger (NLS) equations are a universal model for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and they arise in a variety of physical settings. There are two inequivalent versions of the equation:

$$iq_t + q_{xx} + \sigma 2|q|^2 q = 0\,, \qquad \sigma = \pm 1 \qquad (1)$$

corresponding to the two choices of the relative sign for the nonlinear and the dispersive term. The case $\sigma = -1$ is referred to as the defocusing NLS equation. It describes the stable propagation of an electromagnetic beam in (cubic) nonlinear media with normal dispersion, and has been the subject of renewed applicative interest in the framework of recent experimental observations in Bose-Einstein condensates and dispersive shock waves in optical fibers.

In [1] we addressed two issues in the spectral theory of the scattering problem associated with the defocusing NLS equation, specifically: (i) relate the existence and location of discrete eigenvalues of the scattering problem to the area of the initial profile of the solution, suitably defined to take into account the boundary conditions; (ii) elucidate the radiative contribution to the asymptotic phase difference of the potential.

When $\sigma = 1$ the equation (1) is known as focusing NLS equation, and has been derived in such diverse fields as deep water waves, plasma physics, nonlinear fiber optics with anomalous dispersion, magnetic spin waves, and more.

Our recent interest is in the solution of the initial-value problem for the focusing NLS equation by the Inverse Scattering Transform (IST), when $q(x,t)$ does not approach zero as $x \to \pm\infty$ (non-zero boundary conditions, NZBC).

Even though the IST for the focusing NLS with rapidly decaying potentials was first proposed more than 40 years ago, and has been subsequently the subject of a vast amount of studies and applications, not as much is available in the literature in the case of nontrivial boundary conditions. The reason for this deficiency is twofold: on one hand, the technical difficulties resulting from the NZBC significantly complicate the formulation of the IST; on the other hand, the onset of modulational instability, also known as the Benjamin-Feir instability in the context of water waves, was believed to be an obstacle to the development of the IST, or at least to its validity. Nonetheless, a large number of exact solutions to the focusing NLS equation with NZBC have been found over the years by the use of direct methods. Historically, the first such solution was found by Peregrine in 1983. In recent years these solutions have been actively studied worldwide, and the renewed interest is due to the fact that the development of modulation instability in the governing equation has been recently suggested as a mechanism for the formation of extreme (also known as "rogue", or "freak") waves, where energy density exceeds the mean level by an order of magnitude.

At the same time, the observation of rogue waves has been reported in an optical system, based on a microstructured optical fiber. The generation of these rogue waves has been modelled using a generalized NLS equation, and shown to be an infrequent evolution from initially smooth pulses owing to power transfer seeded by a small noise perturbation.

In view of these recent experimental developments, the investigation of the IST for the focusing case with NZBC presented itself as a very interesting mathematical problem.

In [2] we developed the IST as a tool to solve the initial-value problem for the focusing NLS equation with NZBC $q_{l/r}(t) \equiv A_{l/r} e^{-2iA_{l/r}^2 t + i\theta_{l/r}}$ as $x \to \mp\infty$ in the fully asymmetric case for both asymptotic amplitudes and phases, i.e., with $A_l \neq A_r$ and $\theta_l \neq \theta_r$.

Specifically, we showed that the direct problem is well-defined for NLS solutions $q(x,t)$ such that $\big(q(x,t) - q_{l/r}(t)\big) \in L^{1,1}(\mathbb{R}^{\mp})$ with respect to x for all $t \geq 0$, and established the corresponding analyticity properties of eigenfunctions and scattering data are.

We then formulated the inverse scattering problem both via (left and right) Marchenko integral equations, and as a Riemann-Hilbert problem on a single sheet of the scattering variables $\lambda_{l/r} = \sqrt{k^2 + A_{l/r}^2}$, where k is the usual complex scattering parameter in the IST.

Finally, we derived the time evolution of the scattering coefficients, showing that, unlike the case of solutions with equal amplitudes as $x \to \pm\infty$, here both reflection and transmission coefficients have a nontrivial (although explicit) time dependence.

The results obtained in [2] will be instrumental for the investigation of the long-time asymptotic behavior of fairly general NLS solutions with nontrivial boundary conditions via the nonlinear steepest descent method on the Riemann-Hilbert problem, or via matched asymptotic expansions on the Marchenko integral equations.

2. Coupled Maxwell-Bloch equations with inhomogeneous broadening

The nonlinear interaction between radiation and a multilevel optical medium has received considerable attention over the past decades. The phenomenon that describes the effect of a coherent medium response to an incident electric field, to which the medium is totally transparent and which undergoes lossless propagation, is known as self-induced transparency (SIT). SIT was first discovered by McCall and Hahn in 1968 in the case of a resonant optical media undergoing a pure two-level atomic transition. A large variety of special solutions as well as an infinite number of conservation laws associated with the Maxwell-Bloch equations governing the SIT phenomenon in a two-level medium were found by Lamb in the early seventies. The initial value problem for the propagation of a pulse through a resonant two-level optical medium for the SIT case was solved by applying the IST shortly afterwards. More recently, the IST was employed to solve the Maxwell-Bloch equations in more general setting of two-level unstable optical media to study the superfluorescence phenomenon and related problems in laser optics.

It is also possible to formulate the propagation of optical pulses in a three-level optical medium in the framework of the IST. Optical pulse propagation in a three-level medium under two-photon or double one-photon resonance conditions has been studied extensively theoretically and experimentally by various authors since the 1970s.

The basic physical problem of interest is the propagation of two optical pulses in a medium of three level atoms, in which the excited state $|3\rangle$ decays at a rate Γ to states other than $|1\rangle$ and $|2\rangle$. The electric fields E_1 and E_2 corresponding to the individual optical pulses are resonantly coupled to the $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ atomic transitions, respectively.

The material properties of the optical medium are described by the Bloch density matrix $\tilde{\boldsymbol{\rho}}$, whose diagonal elements are determined by the population densities of the atomic levels, while the off-diagonal elements describe the complex valued material polarizability envelopes of the optical medium. The equations governing the temporal evolution of the atomic levels in the optical medium and the propagation of the optical pulses through the medium can be derived from the Schrödinger and Maxwell's equations using a slowly varying envelope approximation. The resulting system of equations are known as the coupled Maxwell-Bloch (CMB) equations. In the lossless case ($\Gamma = 0$), and under the assumption that the propagation constants, which depend on the dipole moments and the atomic number density for the two optical pulses through the medium, are the same, the resulting CMB equations can be shown to admit a Lax pair, and can be solved by IST.

In [3] we developed the IST to solve the general initial value problem for the CMB equations with inhomogeneous broadening describing the propagation of localized optical pulses decaying sufficiently rapidly as $x \to \pm\infty$, through a three-level medium. Furthermore, these solutions were determined for generic initial preparation of the medium, i.e., for a sufficiently broad class of specified boundary conditions for the Bloch density matrix. Another key issue, not addressed elsewhere before, is to determine the final state of the medium given by the Bloch matrix after the interaction with the electromagnetic field, i.e. the asymptotic value $\tilde{\boldsymbol{\rho}}_+$ of $\tilde{\boldsymbol{\rho}}$ as $x \to \infty$.

Using the IST method we also constructed exact n-soliton solutions for the optical pulses and studied the soliton interaction properties including polarization shifts for the optical pulses.

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