A Model for Staircase Formation in Fingering Convection

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Fingering convection [1] is a convective instability that occurs in fluids where two buoyancy–changing scalars with different diffusivities have a competing effect on density. The peculiarity of this form of convection is that, although the transport of each individual scalar occurs down–gradient, the net density transport is up–gradient. In a suitable range of non–dimensional parameters, solutions characterized by constant vertical gradients of the horizontally averaged fields may undergo a further instability, which results in the alternation of layers where density is roughly homogeneous with layers where there are steep vertical density gradients, a pattern known as doubly–diffusive staircases.

High–resolution, numerical simulations of staircase–forming fingering convection [2] have shown that, as a consequence of the aggregation of small–scale buoyancy–carrying coherent structures into larger–scale clusters [3], the vertical density flux F primarily depends on the vertical density gradient $\bar{\rho}_z$, and that this dependence is non-monotonic. Thus one is tempted to write a closed equation for density having the form

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\partial F_\rho\left(\bar{\rho}_z\right)}{\partial z} = -F'_\rho\left(\bar{\rho}_z\right)\frac{\partial^2 \bar{\rho}}{\partial z^2} \qquad (1)$$

This behaves as a non-linear, but otherwise ordinary, diffusion process where $\bar{\rho}_z$ is such that $F'_\rho(\bar{\rho}_z) < 0$, and it turns into a negative-diffusion process where $\bar{\rho}_z$ is such that $F'_\rho(\bar{\rho}_z) > 0$. One then expects that in the latter case eq. (1) acts so as to increase the fluctuations in the field $\bar{\rho}$, rather than quenching them, as ordinary diffusion does. If $F'_\rho(\bar{\rho}_z) < 0$ for $\bar{\rho}_z$ steeper than a given threshold, then one might naively expect that this simple model could stop the uncontrollable growth of gradients that makes negative diffusion processes ill-posed, because growing fluctuations would generate growing gradients, which eventually would become large enough to be quenched. Unfortunately, this is a deceptive argument. A linear stability analysis shows that eq. (1) undergoes an ultraviolet catastrophe: in a general setting, the problem remains ill-posed.

We avoid the ill–posedness of the model (1) by postulating, on the basis of well–established fluid mechanical theories, such as Prandtl's mixing–length, a dependence of the density flux both on the density gradient and on the kinetic energy of the fluid $\bar{e}$. This leads to a class of models that can be used for expaining staircase–formation phenomena [4]. In particular, we focus our attention on the following two equations:

$$\begin{cases} \bar{b}_t &= -\left(\mathcal{F} - l\bar{e}^{1/2}\bar{b}_z\right)_z \\ \bar{e}_t &= \left(l\bar{e}^{1/2}\,\bar{e}_z\right)_z + \mathcal{F} - l\bar{e}^{1/2}\bar{b}_z - A\bar{e}\bar{b}_z^{1/2} \end{cases} \qquad (2)$$

where $\bar{b}$ is the buoyancy, and it is linked to density by the expression $\rho = \rho_o(1 - g^{-1}b)$ (where ρ_o is a reference density and g is the acceleration of gravity); $\mathcal{F}$ and A are positive free parameters of the problem; and l is the mixing length.

Steady, vertically–uniform solutions of eqs. (2) exhibit the expected non–monotonic dependence of the buoyancy flux on the buoyancy gradient (fig. 1). However, a linear stability analysis proves that the model (2) is immune from the ultraviolet catastrophe. When the unstable infinitesimal perturbation grow and reach the non–linear regime, the vertically–uniform solutions morph into staircase–like profiles (fig. 2).

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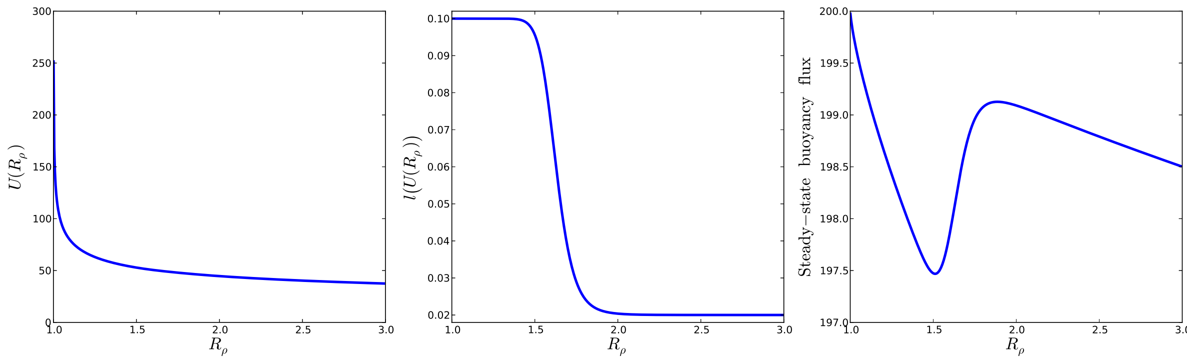

Figure 1. From left to right: velocity scale U, mixing length l, and buoyancy flux $\mathcal{C}$ as a function of the density ratio (that is, the non–dimensional vertical buoyancy gradient) for the steady state solutions of the staircase model. See [5] for the parameters value.

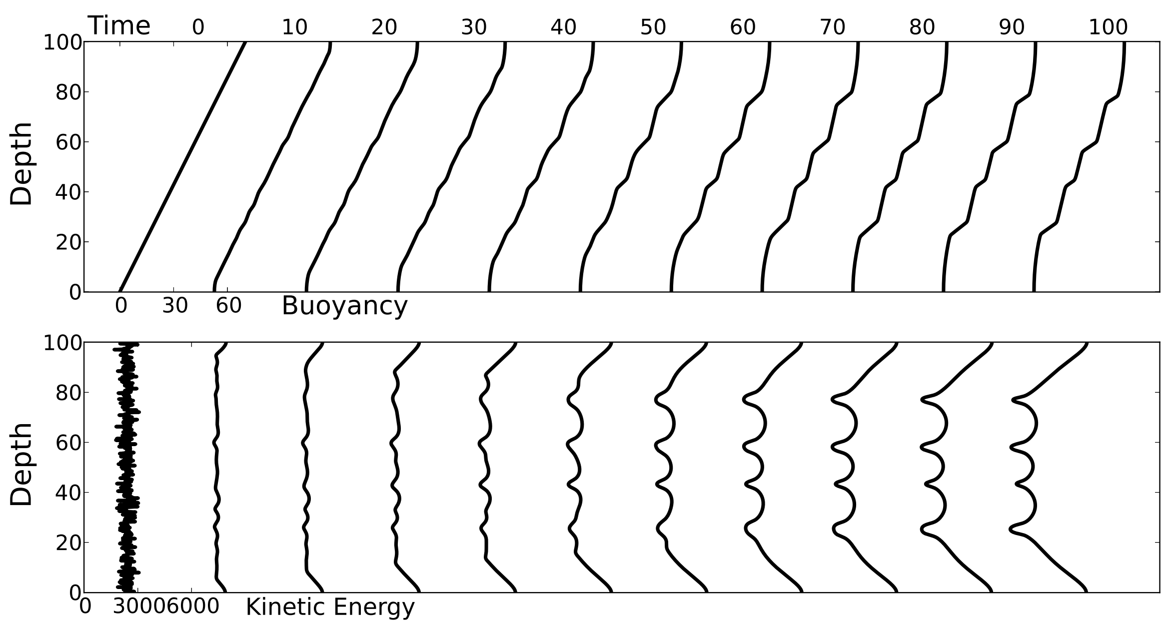

Figure 2. Time evolution of vertical profiles of horizontally averaged buoyancy (upper panel) and kinetic energy density (lower panel) in a numerical solution of equations (2) with $R_\rho = 1.7$ and the parameters of Figure 1. In each panel, the profiles after the first have been shifted to the right as a function of time. All quantities are expressed in arbitrary units.