On the Performance of Mildly Greedy Players in Cut Games

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It has been known since the early fifties that the strategic behavior of selfish players in non-cooperative games usually produces suboptimal outcomes with respect to the ones which could be potentially enforced by a dictatorial authority, the Prisoner’s Dilemma being the most famous and pragmatic example. Nevertheless, it has been only after the seminal paper of Koutsoupias and Papadimitriou [9] in 1999 that this phenomenon, termed as *price of anarchy*, became object of a thorough analytical scrutiny by the scientific community. Formally speaking, given a *social function* measuring the overall quality of all the strategy profiles which can be realized in a game, the price of anarchy measures the *worst-case* ratio between the social value of a a Nash equilibrium and the social value of a strategy profile optimizing the social function (the, so called, *social optimum*).

In the last years, however, a ground-breaking sequence of complexity results has provided a strong evidence of the computational intractability of the problem of computing Nash equilibria in several games of interest. In particular, the problem of computing a pure Nash equilibrium has been shown to be PLS-complete in *congestion games* by Fabrikant *et al.* [8] and in some of their special cases by Ackermann *et al.* [1], where congestion games, introduced by Rosenthal in [11], is a well-known and significative class of games represented in succinct form for which existence of pure Nash equilibria is always guaranteed.

For such a reason, the price of anarchy has to be intended as a **theoretical** bound of inefficiency to which a system populated by selfish agents (what the economists usually call a *market*) may ideally tend to the limit, but which is unlikely to be attained in practice because of computational issues. To explain this situation, Kamal Jain coined the following metaphorical expression: “If your laptop cannot compute it, then neither can the market”. Because of these limitations, in the last years, quite an attention has been moved to the analysis of the performance of less demanding solution concepts, among which are approximate pure Nash equilibria and best-response dynamics of polynomially bounded length.

Approximate pure Nash equilibria are pure Nash equilibria for *mildly greedy players*, that is, players who are willing to be part of any strategy profile in which they experience a utility which is “not too far” from the best utility they can get by deviating to another strategy. More formally, given a value $\epsilon \geq 0$, a $(1+\epsilon)$-approximate pure Nash equilibrium is a strategy profile $\boldsymbol{\sigma}$ such that the utility that each player gets when deviating to any other strategy is no more than $1+\epsilon$ times the utility that she gets in $\boldsymbol{\sigma}$. Any 1-approximate pure Nash equilibrium is a pure Nash equilibrium by definition, hence, the set of pure Nash equilibria is a proper subset of that of $(1+\epsilon)$-approximate pure Nash equilibria for any $\epsilon > 0$. For sufficiently high values of ϵ, the problem of computing a $(1+\epsilon)$-approximate pure Nash equilibrium becomes polynomial time solvable in several games of interest. In particular, there exist polynomial time algorithms for computing one such an equilibrium in several special cases of congestion games (Bhalgat *et al.* [2], Caragiannis *et al.* [4,5], Chien and Sinclair [6]).

A best-response dynamics, instead, is an evolutive process in which, starting from a given strategy profile, the players are processed sequentially and, at each step, each player is allowed to change her current strategy by best-responding to the strategies played by the others. Clearly, when players can compute in polynomial time their best-responses, a best-response dynamics of polynomially bounded length, i.e., with a polynomial number of steps, can be efficiently computed. We speak of an *approximate best-response dynamics* when it involves mildly greedy players. In particular, a $(1+\epsilon)$-approximate best-response dynamics is a dynamics in which each player changes her strategy only when it improves her utility of a factor of more than $1+\epsilon$. By definition, any fixed point of a $(1+\epsilon)$-approximate best-response dynamics is a $(1+\epsilon)$-approximate pure Nash equilibrium. One may define several special cases of best-response dynamics: for instance, Mirrokni and Vetta [10] introduce the notions of covering walks, k-covering walks, one-round walks, k-round walks and random one-round walks. A covering walk is a sequence of best-response dynamics in which each player plays at least once, a k-covering walk is a concatenation of k covering walks, a *one-round walk* is a covering walk in which each player plays exactly once, a k-round walk is a concatenation of k one-round walks, while a random one-round walk is a one-round walk such that the order in which players are processed is chosen randomly. When considering mildly greedy players, the analogous notions of approximate covering walks, approximate k-covering walks, *approximate one-round walks*, and so on, may be defined.

In this paper, we study the performance of mildly greedy players in *cut games*, a relevant subclass of congestion games. Cut games are naturally defined by an undirected edge weighted graph G. Each vertex of G is owned by a player and has to be placed in one of the two possible sides of a bipartition. Each player has to decide which side to choose so as to maximize the sum of the weights of the edges connecting her node to all the nodes belonging to the opposite side. Thus, each strategy profile induces a cut of G and each player wants to maximize the contribution given to the total weight of the cut by the edges incident to her node. The social function mainly used in the literature to measure the overall quality of a strategy profile is the total weight of the induced cut which is half of the sum of the players' utilities.

Each cut game, being a particular instance of congestion games, always admits pure Nash equilibria; moreover, any best-response dynamics is guaranteed to converge to one such an equilibrium in a finite number of steps. However, the computation of one such an equilibrium, being strongly related to that of a local optimum of the MAXCUT problem, is a PLS-complete problem, hence widely believed to be computationally untractable. This justifies the idea of resorting to mildly greedy players who can give life to solutions having a more permissive computational complexity. To this aim, Bhalgat *et al.* [2] give a polynomial time algorithm to compute a $(1+\epsilon)$-approximate pure Nash equilibrium, for any $\epsilon > 2$.

Standard arguments from the theory of approximation algorithms imply that either the price of anarchy and the approximation ratio of the solutions achieved after a one-round walk starting from the empty strategy profile is 1/2 in cut games. Chrisodoulou *et al.* [7] show that a random one-round walk converges to a 1/8-approximation of the social optimum, while, on the negative side, they show that there exist k-round walks converging to an $O(k/n)$-approximation of the social optimum and that there are strategy profiles at exponential distance from any pure Nash equilibrium. Such a worst-case poor deterministic convergence, however, does not occur when mildly greedy players come into play, since they prove that any $(1+\epsilon)$-approximate one-round walk starting from any initial strategy profile converges to a $\left(\min\left\{\frac{1}{4+2\epsilon}, \frac{\epsilon}{4+2\epsilon}\right\}\right)$-approximation of the social optimum.

We give exact bounds on the worst-case performance guarantee of mildly greedy players in cut games by considering either approximate pure Nash equilibria and approximate one-round walks. In particular, we show that the ϵ-approximate price of anarchy, that is the price of anarchy of $(1+\epsilon)$-approximate pure Nash equilibria, is at least $\frac{1}{2+\epsilon}$ and that this bound is tight for any ϵ. We then move to the evaluation of the approximation ratio of the solutions achieved after a $(1+\epsilon)$-approximate one-round walk starting from any initial strategy profile. This notion can be seen as an analogous of the price of anarchy for $(1+\epsilon)$-approximate one-round walks and is defined as the worst-case ratio between the value of a strategy profile realized at the end of the walk and the social optimum. We show that this ratio is at least $\min\left\{\frac{1}{2+\epsilon}, \frac{2\epsilon}{(1+\epsilon)(2+\epsilon)}\right\}$, thus significantly improving the previous lower bound of $\min\left\{\frac{1}{4+2\epsilon}, \frac{\epsilon}{4+2\epsilon}\right\}$ given by Christodoulou *et al.* [7], and prove that also this bound is tight for any ϵ.

Our lower bounds are both obtained by exploiting the primal-dual method introduced by Bilò in [3]. In particular, for the case of approximate one-round walks, a natural but tricky analysis of all the situations which may occur during the walk allows us to exploit the power of the primal-dual method at its full magnitude. Not by chance, in fact, the lower bound that we achieve is much better (at least the double) than the one that could be obtained by Christodoulou *et al.* [7] by making use of combinatorial arguments only.

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