Scale invariant elliptic operators with singular coefficients

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We made a systematic investigation of the operator

$$L = |x|^\alpha \Delta + c|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla - b|x|^{\alpha-2} \tag{1}$$

in $L^p(\mathbb{R}^N)$, $N \geq 1$, $1 < p < \infty$. Here α, b, c are unrestricted real numbers. In particular, in [2], we proved generation results of analytic semigroups, domain characterization and, in [1], we obtained $L^p - L^q$ estimates for the semigroup. We set $\Omega = \mathbb{R}^N \setminus \{0\}$ and define $C_c^\infty(\Omega)$ as the space of infinitely continuously differentiable functions with compact support in Ω. We define L_{min} as the closure in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\Omega))$ and $L_{max} = (L, D_{max}(L))$ where $D_{max}(L)$ is defined as follows

$$\left\{u \in W^{2,p}_{loc}(\Omega) \cap L^p(\mathbb{R}^N) : \ Lu \in L^p(\mathbb{R}^N)\right\}.$$

The domain of L_{min} will be denoted by $D_{min}(L)$. Note that if $u \in D_{max}(L)$ and $f = Lu$ the equation $Lu = f$ is satisfied in the sense of distributions in Ω rather than in $\mathbb{R}^N$. We studied when suitable realizations of L between L_{min} and L_{max} generate a semigroup in $L^p(\mathbb{R}^N)$. The introduction of $C_c^\infty(\Omega)$ instead of $C_c^\infty(\mathbb{R}^N)$ is unavoidable to treat the singularity at 0 but sometimes leads to unnatural difficulties. For example, if $\alpha = b = c = 0$ and $N \geq 3$, then the Laplacian with domain $W^{2,p}(\mathbb{R}^N)$ coincides with Δ_{min} if and only if $p \leq N/2$ and with Δ_{max} if and only if $p \geq N/(N-2)$. Similar problems happen when $C_c^\infty(\mathbb{R}^N) \subset D_{max}(L)$ (depending on α, b, c, p) and this explains why we need also intermediate operators between L_{min} and L_{max}. When $\alpha = c = 0$, L becomes the Schrödinger operator with inverse square potential which is widely studied in the literature. A famous result due to Baras and Goldstein shows that the parabolic equation $u_t = Lu$ presents instantaneous blow-up for positive solutions when $D_0 := b + (N-2)^2/4 < 0$, where $4D_0$ is the discriminant of the quadratic equation

$$f_0(s) := -s^2 + (N-2)s + b = 0.$$

In the general case we show that the elliptic equation $\lambda u - Lu = f$, with $\lambda, f \geq 0$, has no positive solution if $\alpha \neq 2$ and $D_c := b + (N-2+c)^2/4 < 0$. The case $\alpha = 2$ is special in the whole paper and the above restriction is not necessary. We obtained positive results under the assumption $D_c \geq 0$.
In order to formulate our main results we introduce the quadratic function

$$f(s) = -s^2 + (N-2+c)s + b.$$

Its roots are given by

$$s_1 = \frac{N-2+c}{2} - \sqrt{b + \left(\frac{N-2+c}{2}\right)^2},$$

$$s_2 = \frac{N-2+c}{2} + \sqrt{b + \left(\frac{N-2+c}{2}\right)^2}.$$

Note that f has the maximum at $s_0 = (N-2+c)/2$ with $f(s_0) = D_c$.
We proved that if $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (N-2+c)^2/4 > 0$. Then a suitable realization of $L_{min} \subset L_{int} \subset L_{max}$ generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if

$$s_1 + \min\{0, 2-\alpha\} < N/p < s_2 + \max\{0, 2-\alpha\}.$$

In this case the generated semigroup is bounded analytic and positive. The domain of L_{int} is also characterized.

In general the semigroup is not contractive. The case $\alpha = 2$ is special and much simpler: no restriction on N/p is needed.
We observe that L generates a semigroup in some $L^p(\mathbb{R}^N)$ if and only if the open intervals $(s_1 + \min\{0, 2-\alpha\}, s_2 + \max\{0, 2-\alpha\})$ and $(0, N)$ intersect. This is always the case when $b > 0$ since s_1 and s_2 have opposite signs but easy examples show that the contrary can happen if $b \leq 0$. In such cases no realization of L between L_{min} and L_{max} is a generator but it can happen that L endowed with a suitable domain is a generator.

In the critical case $D_c = 0$ we proved that for $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (N-2+c)^2/4 = 0$ and $s_0 = \frac{N-2+c}{2}$, a suitable realization of $L_{min} \subset L_{int} \subset L_{max}$ generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if

$$s_0 + \min\{0, 2-\alpha\} < N/p < s_0 + \max\{0, 2-\alpha\}.$$

In this case the generated semigroup is bounded analytic and positive. A description of the domain of L_{int} is also given.

As next step in the study of this operator, we considered L^p-L^q estimates of the generated semigroup. We focused on the homogeneity property of L, i.e.,

$$L(u(\lambda x)) = \lambda^{2-\alpha}(Lu)(\lambda x),$$

for every $\lambda > 0$ and $x \in \mathbb{R}^N$. We proved precise L^p-L^q estimates for the semigroup generated by L which, in our opinion, are new also for the Schrödinger operator with the inverse square potential. We used the homegeneity property above, the precise description of the domain of L in $L^p(\mathbb{R}^N)$ and the precise range of p for which L is a generator in L^p together with an abstract lemma due to Varopulos. Since, in general, the semigroup generated by operators with unbounded coefficients is not unique, we considered the semigroup generated by L_{int} mentioned above. The description of the domain has been crucial in our approach, providing the embedding of the L^p-domain of L into L^q for p, q sufficiently close; L^p-L^q estimates have been first proved for these p, q and then extended to the whole range of p, q for which there is generation. It is worth mentioning that L^p-L^q estimates never hold, if $p \neq q$, when $\alpha < 0$ or $\alpha = 2$. If $0 \leq \alpha < 2$ it is necessary (but not sufficient) that $p < q$ as for the classical heat equation and the condition $p > q$ is necessary when $\alpha > 2$. In any case, L^p-L^q estimates never hold if p or q falls outside the generation interval.

As a corollary, we deduce the explicit L^p-L^q decay rate for the Schrödinger heat semigroup. We recall that the condition $b \geq -(N-2)^2/4$ is necessary and sufficient for the existence of a positive semigroup. Once this condition is satisfied the real numbers

$$s_1 = \frac{N-2}{2} - \sqrt{b + \left(\frac{N-2}{2}\right)^2},$$

$$s_2 = \frac{N-2}{2} + \sqrt{b + \left(\frac{N-2}{2}\right)^2}$$

are well defined and the Schrödinger operator, with a suitable domain, is a generator in L^r if and only if $s_1 < N/r < s_2 + 2$. We obtained $\|T(t)\|_{p \to q} = Ct^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$ for the generated semigroup if and only if $s_1 < N/q \leq N/p < s_2 + 2$, in the remaining cases $T(t)$ being unbounded form L^p to L^q. Note that in the critical case $b = -(N-2)^2/4$, L^p-L^q estimates hold if and only if $\frac{2N}{N+2} < p \leq q < \frac{2N}{N-2}$.

REFERENCES

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