Nash Stability in Fractional Hedonic Games

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Hedonic games, introduced in [5], are games in which players have preferences over the set of all possible player partitions (called clusterings). In particular, the utility of each player only depends on the composition or structure of the cluster she belongs to. Cluster formation is of fundamental importance in a variety of social, economic, and political problems. Therefore, a big stream of research considered this topic from a strategic cooperative point of view. Nevertheless, studying strategic solutions under a non-cooperative scenario becomes important when considering huge environments (like the Internet) lacking a social planner or where the cost of coordination is tremendously high. In this setting, a clustering is Nash stable if no player can improve her utility by unilaterally changing her own cluster. A non-cooperative research on hedonic games can be found in [6].

A notably class of hedonic games is that of *additively separable* ones [2,4], in which the utility of a player is given by the sum of the weights of the edges being incident to the other players belonging to the same cluster. Moreover, within this class of games, the *symmetric* case, where the weights are given by an undirected edge-weighted graph in which nodes represent players and edge weights measure the happiness of the players for belonging to the same cluster, has received significant attention [3,4].

In this paper, we consider the class of (symmetric) *fractional hedonic games* recently introduced in [1]. The main difference with respect to additive separable hedonic games is that, in the fractional model, the utility of each player in a cluster is divided by the number of players belonging to it. In such a way, fractional hedonic games model natural behavioral dynamics in social environments that are not captured by additive separable ones: one usually prefers having a couple of good friends in a cluster composed by few other people rather than being part of a crowded cluster populated by uninteresting guys. We analyze this class of games from a non-cooperative perspective, with the aim of understanding the existence, computability and performance of Nash stable clusterings.

We first show that in presence of negative edge weights, Nash stable clusterings are not guaranteed to exist, while, if edge weights are non-negative, the basic outcome in which all players belong to the same cluster (*basic Nash stable clustering*) is Nash stable. Then, we evaluate their performance by means of the widely used notions of price of anarchy and price of stability. We give an upper bound of $O(n)$ on the price of anarchy for weighted graphs and show that it is asymptotically tight even for unweighted paths. We also prove a lower bound of $\Omega(n)$ on the price of stability holding even for weighted stars. We observe that, being the basic Nash stable clustering the responsible for such a bad performance, one may ask whether Nash stable clusterings of better quality may exist and be efficiently computed. To this aim, we show that Nash stable clusterings may not be reached by independent selfish agents unless some kind of centralized control is enforced in the game (that is, uncoordinated best-response dynamics may not converge to stable outcomes), even for unweighted bipartite graphs.

This last result, in particular, rises the question of the existence of efficient algorithms for the determination of good quality Nash stable clusterings. To this aim, however, we prove that computing the best quality Nash stable clustering, as well as an optimal (non necessarily stable) one, is an NP-hard problem. Given the above negative and impossibility results, we focus on fractional hedonic games played on particular graph topologies such as unweighted bipartite graphs and unweighted trees which already pose challenging questions and require non-trivial approaches. Fractional Hedonic games played on unweighted graphs have natural application to politics and social groups [1]. For bipartite graphs we show that the price of stability is strictly greater than 1 and

provide a polynomial time algorithm computing a Nash stable clustering approximating the social optimum by a factor strictly smaller than 2 (thus proving that 2 is an upper bound to the price of stability in this setting). For trees, we prove that the price of stability is 1 and show how to constructively compute in polynomial time an optimal Nash stable clustering.

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