Standard monomial theory for wonderful varieties

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The first appearance of the idea of a standard monomial theory may be traced back to Hodge's study of Grassmannians in [11], [12]. Then Doubilet, Rota and Stein found a basis with similar properties for the coordinate rings of the space of matrices in [9]. This was reproved and generalized to the space of symmetric and antisymmetric matrices by De Concini and Procesi in [7].

A systematic program for the development of a standard monomial theory for quotient of reductive groups by parabolic subgroups was then started by Seshadri in [19] where the case of minuscule parabolics is considered. Further, in [14] Seshadri and Lakshmibai noticed that the above recalled previous results could be obtained as specializations of they general theory.

This program was finally completed by Littelmann. Indeed, in [15], he found a combinatorial character formula for representations of symmetrizable Kac-Moody groups introducing the language of L-S paths. Moreover, he used this as an index set for the basis constructed in [16] and he proved that this basis defines a standard monomial theory for Schubert varieties of symmetrizable Kac-Moody groups. This theory has been developed in the context of LS algebras over poset with bonds in [2], [3] and [4].

We want now to briefly recall what a standard monomial theory is, the reader may see [5] for further details about this general setting. Let $\mathbb{A}$ be a finite subset of an algebra A and suppose we are given a transitive antisymmetric binary relation $\longleftarrow$ on $\mathbb{A}$. We define an abstract monomial $\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ of elements of $\mathbb{A}$ as standard if $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$. If the set of standard monomials is a basis of the algebra A as a vector space then we say that $(\mathbb{A},\longleftarrow)$ is a standard monomial theory for A. Suppose, further, we have a monomial order $\leqslant_t$ on the monomials of elements of $\mathbb{A}$. By the previous assumption, we may write any non-standard monomial m' as a linear combination of standard monomials. If in such an expression only standard monomials m with $\mathsf{m}' \leqslant_t \mathsf{m}$ appear, then we say that we have a straightening relation for m'. If we have a straightening relation for each non-standard monomial, then we say that $(\mathbb{A},\longleftarrow,\leqslant_t)$ is a standard monomial theory with straightening relations.

Given a simply connected semisimple algebraic group G over an algebraically closed field $\Bbbk$ of characteristic 0, a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, let $\Lambda^+ \subset \Lambda$ be the monoid of dominant weights and the lattice of weights, respectively. Fix a dominant weight $\lambda \in \Lambda^+$ with stabilizer a parabolic $B \subset P \subset G$, denote by $\mathbb{B}_\lambda$ the set of L-S paths of shape λ

and let V_λ be the G–irreducible module of highest weight λ. Littelmann's construction provide a basis $\mathbb{A}_\lambda = \{p_\pi \,|\, \pi \in \mathbb{B}_\lambda\}$, indexed by L-S paths, for the module $\Gamma(G/P, \mathcal{L}_\lambda) \simeq V_\lambda^*$, where $\mathcal{L}_\lambda$ is the line bundle over G/P associated to λ. The multiplication of sections $\Gamma(G/P, \mathcal{L}_\lambda)^{\otimes 2} \longrightarrow \Gamma(G/P, \mathcal{L}_{2\lambda})$ induces a ring structure on the algebra $A(G/P) = \bigoplus_{n \geqslant 0} \Gamma(G/P, \mathcal{L}_{n\lambda})$. This algebra is the coordinate ring of the cone over the embedding of $G/P \hookrightarrow V_\lambda$ induced by $\mathcal{L}_\lambda$. On the basis $\mathbb{A}_\lambda$ one may define a relation $\longleftarrow$ and a monomial order $\leqslant_t$ such that $(\mathbb{A}_\lambda, \longleftarrow, \leqslant_t)$ is a standard monomial theory with straightening relations for $A(G/P)$.

In [6], the second and fourth named authors adapted Littelmann's basis to the Cox ring (see below) of complete symmetric varieties; this class of varieties has been introduced by C. De Concini and C. Procesi in [8]. As a result, they proved the degeneration of the Cox ring to the coordinate ring of a suitable multicone over a flag variety. This degeneration allowed a new proof of the rational singularity property for the Cox ring of complete symmetric varieties.

The purpose of the present paper is a further extension of these results to the Cox ring of wonderful varieties. As a first step, we use the occasion to introduce a general setting for a multigraded standard monomial theory modelled on the above recalled one. This setting may be briefly summarized as follows.

Let $\mathbb{A} \doteq \mathbb{A}_1 \cup \mathbb{A}_2 \cup \cdots \cup \mathbb{A}_n$ be the union of finite subsets of an algebra A. Suppose we have a binary relation $\longleftarrow$ on $\mathbb{A}$ such that $\longleftarrow$ restricted to $\mathbb{A}_i$ is transitive and antisymmetric for all $i = 1, 2, \ldots, n$ and, further, suppose we have bijective maps $\phi_{i,j}$ from the set of comparable pairs $\mathsf{a} \longleftarrow \mathsf{b}$ of $\mathbb{A}_i \times \mathbb{A}_j$ to the set of comparable pairs $\mathsf{a}' \longleftarrow \mathsf{b}'$ of $\mathbb{A}_j \times \mathbb{A}_i$ satisfying some mild conditions. We define a formal monomial $\mathsf{a}_1\mathsf{a}_2 \cdots \mathsf{a}_N$ as weakly standard if $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$, and we say it is standard if all monomials obtained by swapping in all possible ways adjacent pairs are weakly standard. We define the multigrade of a monomial $\mathsf{a}_1\mathsf{a}_2 \cdots \mathsf{a}_N$ as $(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ where k_i is the number of elements of $\mathbb{A}_i$ in the monomial. If the set of standard monomials is a basis for A as a vector space and this basis induces a multigrading for A, we say that $(\mathbb{A}, \longleftarrow, \phi_{i,j})$ is a multigraded standard monomial theory for A. As above we introduce also a monomial order and the straightening relations for non-standard monomials.

As a motivating example for this setting one may see the multigraded standard monomial theory for the multicone over a flag variety constructed by the second named author in [4].

Now we recall what are the type of varieties we are interested in. A G–variety X is wonderful of rank r if it satisfies the following conditions:

- X is smooth and projective;
- X possesses an open orbit whose complement is a union of r smooth prime divisors, called the boundary divisors, with non-empty transversal intersections;
- any orbit closure in X equals the intersection of the prime divisors which contain it.

Examples of wonderful varieties are the flag varieties, which are the wonderful varieties of rank zero, and the complete symmetric varieties. Wonderful varieties have been considered in full generality by D. Luna in [17], [18] in the context of spherical varieties. See [1] for a general introduction to wonderful varieties.

If X is a wonderful G–variety, then the Picard group $\mathrm{Pic}(X)$ is freely generated by the classes of the B–stable prime divisors of X which are not G–stable. These divisors are called the colors of X. Since X contains an open B–orbit, as in the case of the Schubert divisors of a flag variety, the colors form a finite set Δ, so that $\mathrm{Pic}(X)$ is a free lattice of finite rank.

Given $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(X)$, we have a multiplication map

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{L}') \longrightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{L}').$$

This induces an algebra structure on the direct sum

$$C(X) \quad \doteq \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X)} \Gamma(X, \mathcal{L}),$$

which is called the Cox ring of X.

Denote by $\sigma_1, \ldots, \sigma_r$ the boundary divisors of X, and let s_i be the canonical section of σ_i, for $i = 1, \ldots, r$. As an algebra $C(X)$ is generated by the sections of the line bundles $\mathcal{L}_D$ with $D \in \Delta$ together with the sections $s_1, \ldots, s_r$.

Given $D \in \mathbb{Z}\Delta$, we denote by $\mathcal{L}_D$ the corresponding line bundle. By definition, X contains a unique closed G–orbit $Y \simeq G/P$, and given $D \in \mathbb{N}\Delta$ we denote by λ_D the highest weight of the dual of the simple G–module $\Gamma(Y, \mathcal{L}_D\big|_Y)$, so that $\mathcal{L}_D\big|_Y \simeq \mathcal{L}_{\lambda_D}$ corresponds to the equivariant line bundle on G/P associated to the dominant weight λ_D. By taking into account the description of $\Gamma(X, \mathcal{L}_D)$ as a G–module, we lift Littelmann's basis of $\Gamma(Y, \mathcal{L}_{\lambda_D})$ to X, and we take as algebra generators for $C(X)$ this set of lifts together with the sections $s_1, \ldots, s_r$.

Let $\Delta = \{D_1, \ldots, D_q\}$ and consider the coordinate ring

$$A(G/P) = \bigoplus_{(n_1,\ldots,n_q)\in\mathbb{N}^q} \Gamma(G/P, \mathcal{L}_{n_1\lambda_{D_1}+\cdots+n_q\lambda_{D_q}})$$

of the multicone over the flag variety $Y \simeq G/P$ associated to the dominant weights $\lambda_{D_1}, \ldots \lambda_{D_q}$. We extend in a natural way the multigraded standard monomial structure of $A(G/P)$ to $C(X)$.

As a consequence of our standard monomial theory, we obtain a flat deformation which degenerates $C(X)$ to the product $\Bbbk[s_1, \ldots, s_r] \otimes A(G/P)$. Since multicones over flag varieties have rational singularities by [13] and since rational singularities are stable under deformation by [10], it follows that the Cox ring $C(X)$ has rational singularities as well.

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