Group algebras whose symmetric elements are Lie metabelian

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Let R be a ring with involution $*$. We denote by R^+ the set of symmetric elements of R; namely, $R^+ = \{r \in R | r^* = r\}$. A general problem of interest is to decide if crucial information on the algebraic structure of R can be deduced from properties of R^+. For instance, in this direction, a celebrated result of Amitsur states that if R^+ satisfies a polynomial identity, then so does R.

Let FG be the group ring of a group G over a field F of characteristic $p \neq 2$. If $* : G \longrightarrow G$ is an involution, then it can be extended linearly to an involution on FG, also denoted by $*$. In this case, $(FG)^+$ is the set of linear combinations of terms of the form $g + g^*$, $g \in G$. If $g^* = g^{-1}$ for all $g \in G$, then the induced involution on FG is called the classical involution. Over the past two decades, a lot of attention has been devoted to determining if Lie identities satisfied by $(FG)^+$ are also satisfied by the whole group ring. Along this line, Giambruno and Sehgal [2] showed that if $(FG)^+$ is Lie nilpotent, and G has no 2-elements, then so is FG. Lee proved a similar result for the bounded Lie Engel property in [4], and Lee, Sehgal and Spinelli [6] investigated the question of when $(FG)^+$ is Lie solvable, under suitable restrictions on the orders of the elements of G. Results for arbitrary groups have been proved as well, and we refer the reader to [[5], Chapter 3] for an overview.

More recently, a considerable amount of work involving other involutions on G has appeared. In this setting, Jespers and Ruiz Marn ([3]) determined when $(FG)^+$ is commutative. Subsequently, in Giambruno, Polcino Milies and Sehgal [1] and Lee, Sehgal and Spinelli [7], the conditions under which $(FG)^+$ is Lie nilpotent or bounded Lie Engel were established. In particular, if G has no elements of order 2, then the same result holds as for the classical involution.

On any ring R, let $[x_1, x_2] = x_1x_2 - x_2x_1$. We say that a subset S of R is Lie metabelian if $[[s_1, s_2], [s_3.s_4]] = 0$ for all $s_i \in S$. The purpose of this paper is to examine the conditions under which the set of symmetric elements (with respect to the linear extension of an arbitrary group involution) of FG is Lie metabelian, where G is a torsion group without 2-elements. Our main task is to prove the following result.

Theorem A *Let F be a field of characteristic different from 2 and G a finite group of odd order having an involution $*$. Let FG have the induced involution. If $(FG)^+$ is Lie metabelian, then G is nilpotent.*

For the classical involution, Levin and Rosenberger proved that if G is finite and has no elements of order 2 or 3, then $(FG)^+$ is Lie metabelian if and only if G is abelian. Using Theorem A, we can generalize their result. Indeed, we deduce that if $p \in \{2, 3\}$, then the prohibition of 3-elements in G can be removed, and we can allow for an arbitrary involution on G.

Theorem B *Let F be a field of characteristic $p \in \{2, 3\}$ and G a torsion group with involution having no elements of order 2. Extend the involution linearly to FG. Then $(FG)^+$ is Lie metabelian if and only if G is abelian.*

Details of the work can be found in Ref. [9].

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