On fixed points of central automorphisms of finite–by–nilpotent groups

F. Catino, F. de Giovanni, M.M. Miccoli, [1]

[1]Dipartimento di Matematica e Fisica, Università del Salento, Italy

An automorphism α of a group G is called a *central automorphism* if it acts trivially on the centre factor group $G/Z(G)$, or equivalently if it commutes with all inner automorphisms of G. Central automorphisms of a group G form a normal subgroup $Aut_c(G)$ of the full automorphism group $Aut(G)$ of G. Obviously, the identity is the unique central automorphism of any group with trivial centre, while all automorphisms of an abelian group are central; more in general, nilpotent groups are rich of central automorphisms. The behaviour of central automorphisms has been investigated by several authors (see for instance [1],[2],[3],[4],[7]), and it turns out that such automorphisms play a relevant role in many problems concerning nilpotent groups.

Let G be a group, and let $K(G)$ be the set of all elements x of G such that $x^\alpha = x$ for every central automorphism α of G. Then $K(G)$ is a characteristic subgroup of G, that will be called the *central kernel* of G. It is easy to show that every central automorphism of a group G fixes all elements of the commutator subgroup G' of G, so that G' is contained in $K(G)$ for any group G. On the other hand, the central kernel of a nilpotent group G is often larger than G'. In fact, let p be an odd prime, and consider the semidirect product $G = \langle y\rangle \ltimes \langle x\rangle$, where x has order p^3, y has order p^2 and $x^y = x^{1+p}$; then $Z(G)$ has order p and G' has order p^2, while the central kernel has order p^3. It was also remarked by H. Liebeck [5] that if G is any p-group whose centre has exponent p, then $K(G)$ contains the subgroup G^p generated by all p-powers of elements of G; in particular, $K(G)$ coincides with the Frattini subgroup $\Phi(G)$, if G is a finite p-group and its centre has exponent p (with the obvious exception $|G| = 2$).

The relevance of the subgroup $K(G)$ was already remarked by M.R. Pettet [6], and the aim of this paper is to study finite-by-nilpotent groups with large central kernel (recall here that a group G is *finite-by-nilpotent* if there is a positive integer n such that $\gamma_n(G)$ is finite). Our first main result is the following.

Theorem A *Let G be a finite-by-nilpotent group such that the index $|G : K(G)|$ is finite. Then the subgroup consisting of all elements of finite order of G is finite.*

A slight modification of the above example shows that in the statement of Theorem A one cannot expect that the group G must be finite, even when it is finitely generated. Let p be an odd prime, and $G = \langle y\rangle \ltimes \langle x\rangle$, where x has order p^k for some integer $k \geq 2$, a has infinite order and $x^a = x^{1+p}$; then G is an infinite nilpotent group with finite commutator subgroup, but $K(G)$ has finite index in G. On the other hand, the main obstacle here seems to be the fact that periodicity is not inherited from the factor group $G/K(G)$ to the group G. In fact, Theorem A has a direct consequence which can be considered as an improvement - for periodic groups - of the well-known result by P. Hall on the finiteness of nilpotent groups in which the commutator subgroup has finite index.

Corollary *Let G be a periodic finite-by-nilpotent group such that the index $|G : K(G)|$ is finite. Then G is finite.*

Recall that a group G is called a *Černikov group* it it is abelian-by-finite and satisfies the minimal condition on subgroups. It is well-known that if G is a nilpotent group and G/G' is a Černikov group, then G itself is a Černikov group. Thus our second main result provides a further evidence of the fact that the commutator subgroup and the central kernel of a periodic nilpotent group behave similarly.

Theorem B *Let G be a periodic finite-by-nilpotent group such that $G/K(G)$ is a Černikov group. Then G is a Černikov group.*

It is also known that if G is any nilpotent group such that G/G' is a π-group for some set π of prime numbers, then G is likewise a π-group. We will prove that a corresponding result holds for a periodic nilpotent group G, when the commutator subgroup is replaced by the central kernel $K(G)$, at least for odd prime numbers.

Details of the work can be found in Ref. [8].

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