Split Strongly Abelian p-Chief Factors and First Degree Restricted Cohomology

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Let p be an arbitrary prime number, and let G be a finite group whose order is divisible by p. Moreover, let $\mathbb{F}_p[G]$ denote the group algebra of G over the field $\mathbb{F}_p$ with p elements, and let S be an irreducible (unital left) $\mathbb{F}_p[G]$-module. Then $[G:S]_{\mathrm{p-split}}$ denotes the number of *p-elementary abelian chief factors* or for short *p-chief factors* G_j/G_{j-1} $(1 \le j \le n)$ of a given chief series

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

that are isomorphic to S as $\mathbb{F}_p[G]$-modules and for which the exact sequence $\{1\} \to G_j/G_{j-1} \to G/G_{j-1} \to G/G_j \to \{1\}$ splits in the category of groups. It is well known that $[G:S]_{\mathrm{p-split}}$ is independent of the choice of the chief series of G (see also Theorem 2 below).

W. Gaschütz proved the "only if"-part of the following result on split (or complementable) p-chief factors of finite p-solvable groups (see [3, Theorem VII.15.5]). The converse of Gaschütz' theorem is due to U. Stammbach [6, Corollary 1]), and in an equivalent form it was already proved earlier by W. Willems [7, Theorem 3.9].

Theorem 1. *A finite group G is p-solvable if, and only if, $\dim_{\mathbb{F}_p} H^1(G,S) = \dim_{\mathbb{F}_p} \mathrm{End}_{\mathbb{F}_p[G]}(S) \cdot [G:S]_{\mathrm{p-split}}$ holds for every irreducible $\mathbb{F}_p[G]$-module S.*

Let $C_G(M) := \{g \in G \mid g \cdot m = m \text{ for every } m \in M\}$ denote the *centralizer* of an $\mathbb{F}_p[G]$-module M in G. In order to be able to apply his cohomological characterization of p-solvable groups (see [5, Theorem A]) in the proof of Theorem 1, Stammbach established the following result (see the main result of [6]):

Theorem 2. *Let G be a finite group, and let S be an irreducible $\mathbb{F}_p[G]$-module with centralizer algebra $\mathbb{D} := \mathrm{End}_{\mathbb{F}_p[G]}(S)$. Then*

$$[G:S]_{\mathrm{p-split}} = \dim_{\mathbb{D}} H^1(G,S) - \dim_{\mathbb{D}} H^1(G/C_G(S),S)$$

holds. In particular, $[G:S]_{\mathrm{p-split}}$ is independent of the choice of the chief series of G.

The goal of this paper is to investigate whether analogues of Theorem 1 and Theorem 2 hold in the context of restricted Lie algebras. Recently, the authors have obtained analogues of these results for ordinary Lie algebras (see [1, Theorem 4.3] and [1, Theorem 2.1], respectively). The key result of this paper is a restricted analogue of Theorem 2. All the other major results in this paper are consequences of it and [1, Theorem 5.5]. Let us remark that the characterizations of solvable restricted Lie algebras by the cohomological and representation-theoretic properties of this paper ultimately follow from [1, Theorem 4.3]. Contrary to group algebras of finite groups, universal enveloping algebras of non-zero finite-dimensional Lie algebras are infinite-dimensional. Therefore, the proof of [1, Theorem 4.3] requires filtration techniques. It would have been possible to make this paper independent of [1] by using (co-)induced modules for restricted universal enveloping algebras instead of truncated (co-)induced modules for ordinary universal enveloping algebras.

As a consequence of our results and the equivalence (i)$\Longleftrightarrow$(iv) in [1, Theorem 5.5], we obtain the analogue of Theorem 1 for split strongly abelian p-chief factors of restricted Lie algebras. In the final section we apply the results obtained in Section 2 to the second Loewy layer of the projective cover of the trivial irreducible module.

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