FREE GRADIENT DISCONTINUITY AND IMAGE INPAINTING

Michele Carriero [a], Antonio Leaci [a] and Franco Tomarelli [b]

[a]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Italy

[b]Politecnico di Milano, Dipartimento di Matematica "Francesco Brioschi", Italy

In image restoration the term inpainting denotes the process of filling the missing information in subdomains where a given image is damaged: these domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or aging of canvas and colors in a painting ([3], [4], [15]).
Minimization of Blake & Zisserman functional is a variational approach to segmentation and denoising in image analysis which deals with free discontinuity, free gradient discontinuity and second derivatives: this second order functional was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other drawbacks which occur in lower order models as in case of Mumford & Shah functional (see Ref. [18]). We refer to [1], [15], [9], for motivation and analysis of variational approach to image segmentation and digital image processing.
In this paper we face the inpainting problem for a monochromatic image with a variational approach: solving a Dirichlet type problem for the main part of Blake & Zisserman functional. A similar problem was studied in [11] with the aim of finding a segmentation of a given noisy image.
Mumford & Shah model has been adapted by several authors to the inpainting problem, but some inconvenient has been detected in this approach (see Ref. [15]). In the Mumford & Shah model (see Ref. [17], [18]), the preferable edge curves are those which have the shortest length, therefore it favours straight edges and it produces the emerging of artificial corners. In the Blake & Zisserman model, the presence of second derivatives smooths such corners.
About minimization of the Blake & Zisserman functional under Neumann boundary condition we refer to [8]. For a description of the rich list of differential, integral and geometric extremality conditions we refer to [9]. The results of the paper [11] were deeply exploited in [10] and [12] to study fine properties of local minimizers of Blake & Zisserman functional under Neuman boundary condition, particularly about their singular set related to optimal segmentation; in the present paper they are applied to the derivation and study of a variational algorithm for image inpainting.

In this paper we propose two different second order functionals E^δ and F^δ to deal with image inpainting. The two functionals respectively focus on the cases of complete or partial loss of information in a small subregion.
First we focus on the functional E, which is defined as follows:

$$E(K_0, K_1, v) = \int_{\Omega\setminus(K_0\cup K_1)} \left|D^2 v\right|^2 \, d\mathbf{x} + \alpha\mathcal{H}^1\left(K_0 \cap \overline{\Omega}\right) + \beta\mathcal{H}^1\left((K_1 \setminus K_0) \cap \overline{\Omega}\right) . \tag{1}$$

To face the inpainting problem we look for minimizers of $E^\delta = E(K_0, K_1, v) + \delta\int_\Omega |v|^2 \, d\mathbf{x}$, with $\alpha, \beta, \delta > 0$, among admissible triplets (K_0, K_1, v), say triplets fulfilling

$$\begin{cases} K_0\,,\, K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed,} \\ v \in C^2\left(\widetilde{\Omega} \setminus (K_0 \cup K_1)\right),\ v \text{ approximately continuous in } \widetilde{\Omega} \setminus K_0, \\ v = w \text{ a.e. in } \widetilde{\Omega} \setminus \overline{\Omega}\,, \end{cases} \tag{2}$$

where $\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2$ are open sets, Ω with piecewise C^2 boundary and w is a given function in $\widetilde{\Omega} \setminus \overline{\Omega}$.
The raw image under processing is damaged due to the presence of blotches in the set $\overline{\Omega}$: the noiseless brightness intensity w of the image is known in $\widetilde{\Omega} \setminus \overline{\Omega}$ while is completely lost in the possibly disconnected set Ω.
If (K_0, K_1, u) is a minimizing triplet of E^δ, then u provides the inpainted restoration of the whole image, and $K_0 \cup K_1$ can be interpreted as an optimal segmentation of the restored image: the three elements of a minimizing triplet (K_0, K_1, u) play respectively the role of edges, creases and smoothly varying intensity in the region $\widetilde{\Omega} \setminus (K_0 \cup K_1)$ for the segmented image.

Our result for monochromatic images is stated below in Theorem 1 in the simplified case when the image is smooth where damage does not occur.
About RGB color images, we refer to a forthcoming paper ([14]).

Theorem 1. *Let α, β, δ, Ω, $\widetilde{\Omega}$ and w be s.t.*

$$0 < \beta \le \alpha \le 2\beta,\ \ \delta > 0 \tag{3}$$

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2\,, \tag{4}$$

$$\Omega \text{ is an open set with piecewise } C^2 \text{ boundary } \partial\Omega\,,\ \widetilde{\Omega} \text{ is an open set,} \tag{5}$$

$$w \text{ has a } C^2(\widetilde{\Omega}) \text{ extension which fulfils } \ D^2 w \in L^\infty(\widetilde{\Omega}\,)\,. \tag{6}$$

Then there exists a triplet (C_0, C_1, u) which minimizes the functional

$$E^\delta(K_0, K_1, v) := E(K_0, K_1, v) + \delta \int_\Omega |v|^2\, d\mathbf{x} \tag{7}$$

with finite energy, among admissible triplets (K_0, K_1, v) according to (1), (2),
Moreover any minimizing triplet (K_0, K_1, v) fulfils:

$$K_0 \cap \overline{\Omega} \text{ and } K_1 \cap \overline{\Omega} \text{ are } (\mathcal{H}^1, 1) \text{ rectifiable sets,} \tag{8}$$

$$\mathcal{H}^1(K_0 \cap \overline{\Omega}) \;=\; \mathcal{H}^1(\overline{S_v})\,, \quad \mathcal{H}^1(K_1 \cap \overline{\Omega}) \;=\; \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v)\,, \tag{9}$$

$$\left\{ \begin{array}{c} v \in GSBV^2(\widetilde{\Omega}),\ \textit{hence} \\ v \text{ and } \nabla v \text{ have well defined two-sided traces, finite } \mathcal{H}^1 \text{ a.e. on} \, K_0 \cup K_1\,, \end{array} \right. \tag{10}$$

where S_v and $S_{\nabla v}$ respectively denote the singular sets of v and ∇v.

The main result of this paper is in Theorem 2 (which is not reported here): the statement is quite technical but it is a more useful tool than Theorem 1, since it deals with discontinuity and gradient discontinuity in $\widetilde{\Omega} \setminus \Omega$ of the given raw image w to be processed, together with some additional noisy information denoted by g in a Borel subset $\Omega \setminus U$, where

$$U \subset\subset \Omega \subset\subset \widetilde{\Omega}. \tag{11}$$

Theorem 2 provides the existence of minimizers for the second functional proposed in this paper, which is labeled with F^δ and deals with the noisy part of the image adding a fidelity term to the functional E^δ. Precisely, we set

$$F^\delta(K_0, K_1, v) = E^\delta(K_0, K_1, v) + \mu \int_{\Omega \setminus U} |v - g|^2\, d\mathbf{x} \tag{12}$$

and we look for minimizers of $F^\delta(K_0, K_1, v)$ among triplets (K_0, K_1, v) verifying (2). We apply direct methods of Calculus of Variations to functional (12) by proving the partial regularity for solutions of a weak version $\mathcal{F}^\delta$ of (12).

We emphasize that if (K_0, K_1, v) is a minimizing triplet of F^δ than v fulfils the Euler equations

$$\Delta^2 v + \mu(v - g) = 0 \qquad \text{in } \ \Omega \setminus (\overline{U} \cup K_0 \cup K_1), \tag{13}$$

$$\Delta^2 v + \delta v = 0 \qquad \text{in } \ U \setminus (K_0 \cup K_1) \tag{14}$$

together with many kind of integral and geometric relationships as like as minimizing triplet of Blake & Zisserman functional for image segmentation (see [9], [12]).
To achieve the existence of minimizing triplets of F^δ, inspired by the seminal paper of De Giorgi and Ambrosio [16], we introduce a relaxed functional: the weak Blake & Zisserman functional for inpainting $\mathcal{F}^\delta(v)$. The idea is to deal with a simpler object, just depending on the function v, and then to recover the set of jumps K_0 and creases $K_1 \setminus K_0$ by taking respectively the discontinuity set $\overline{S_v}$ and $\overline{S_{\nabla v}} \setminus S_v$. The functional class where we set the problem is given by second order generalized functions with special bounded variation: say $GSBV^2(\widetilde{\Omega})$. The class $GSBV^2(\widetilde{\Omega})$ is the right functional setting, more appropriate than $BH(\widetilde{\Omega})$ (bounded hessian functions whose second derivatives are Radon measure). Indeed BH

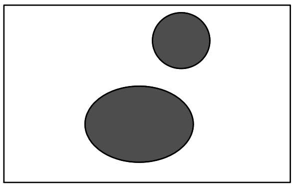

Figure 1. Theorem 1: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with complete loss of information are the black region $\overline{\Omega}$.

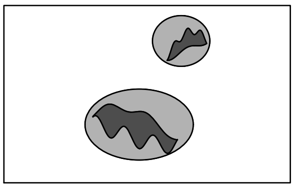

Figure 2. Theorem 2: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with some loss of information, complete loss of information in the black region U, the partially damaged image is given in the gray region $\Omega \setminus U$.

functions in two variables are continuous with integrable gradient; nevertheless BH contains too much irregular functions: admissible functions may have gradient with nontrivial Cantor part.
In this framework compactness and lower semicontinuity Theorems 8 and 10 of [7] give the existence of minimizers for the relaxed functional $\mathcal{F}^{\delta}(v)$. The results of Theorems 1 and 2 are achieved by showing partial regularity of the obtained weak solution with penalized Dirichlet datum. The novelty here consists in the regularization at the boundary for a free gradient discontinuity problem with Dirichlet datum (in the set $\partial\Omega$) or transmission condition (in the set ∂U).

A numerical scheme, based on the theory of Γ-convergence, as in [2] and [5], the convergence analysis and its implementation are contained in a forthcoming paper [13].

We conclude by showing some pictures obtained in numerical experiments which exploit the variational approximation of the functional (12): Figures 3 and 4 where the inpainting algorithm removes masks or overlapping text.

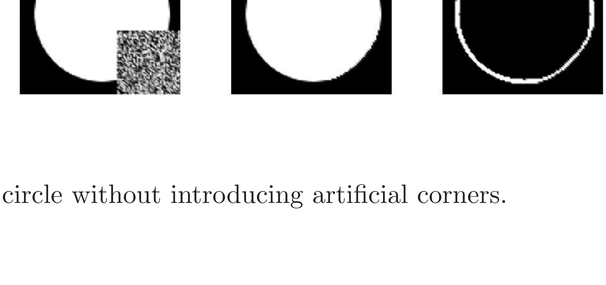

Figure 3. Inpainting of a circle without introducing artificial corners.

Figure 4. Text removal.

REFERENCES

1. L.Ambrosio, N.Fusco & D.Pallara, *Functions of Bounded Variation and Free Discontinuity Problems,* Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.
2. L.Ambrosio, L.Faina & R.March, *Variational approximation of a second order free discontinuity problem in computer vision,* SIAM J. Math. Anal., **32** (2001), 1171–1197.
3. G.Aubert & P.Kornprobst, *Mathematical problems in image processing, Partial Differential Equations and the Calculus of Variations,* Applied Mathematical Sciences, **147** 2nd ed., Springer, New York, 2006.
4. M.Bertalmío, V.Caselles, S.Masnou & G.Sapiro, Inpainting, in "Encyclopedia of Computer Vision", Springer, 2011.
5. A.Braides, A.Defranceschi & E.Vitali, *A compactness result for a second-order variational discrete model,* ESAIM: Mathematical Modelling and Numerical Analysis, **46** (2012), 389-410.
6. M.Carriero & A.Leaci, *Existence theorem for a Dirichlet problem with free discontinuity set,* Nonlinear Analysis, Th. Meth. Appl., **15**, n.7 (1990), 661–677.
7. M.Carriero, A.Leaci & F.Tomarelli, *A second order model in image segmentation: Blake & Zisserman functional,* in "Variational Methods for Discontinuous Structures" (Como, 1994), Progr. Nonlinear Differential Equations Appl. **25**, Birkhäuser, Basel, (1996), 57–72.
8. M.Carriero, A.Leaci & F.Tomarelli, *Strong minimizers of Blake & Zisserman functional,* Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), **25**, n.1-2 (1997), 257–285.
9. M.Carriero, A.Leaci & F.Tomarelli, *Euler equations for Blake & Zisserman functional,* Calc. Var. Partial Differential Equations, **32**, n.1 (2008), 81–110.
10. M.Carriero, A.Leaci & F.Tomarelli, *Uniform density estimates for Blake & Zisserman functional,* Discrete Contin. Dyn. Syst., Series A, **31**, n.3 (2011).
11. M.Carriero, A.Leaci & F.Tomarelli, *A Dirichlet problem with free gradient discontinuity,* Advances in Mathematical Sciences and Applications, **20**, n.1 (2010), 107–141.
12. M.Carriero, A.Leaci & F.Tomarelli, *A candidate local minimizer of Blake & Zisserman functional,* J. Math. Pures Appl., **96,** (2011), 58–87.
13. M. Carriero, A. Leaci & F. Tomarelli *Variational approximation of a second order free discontinuity problem for inpainting,* to appear on Advances in Calculus of Variations, (2013).
14. M. Carriero, A. Leaci & F. Tomarelli *Segmentation and inpainting of color images,* to appear, (2013).
15. T.F.Chan & J.Shen, *Variational image inpainting,* Comm. Pure Appl. Math., **LVIII** (2005), 579–619.
16. E. De Giorgi & L.Ambrosio, *Un nuovo tipo di funzionale del calcolo delle variazioni,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **82** (1988), 199–210.
17. E. De Giorgi, M.Carriero & A.Leaci, *Existence theorem for a minimum problem with free discontinuity set,* Arch. Rational Mech. Anal., **108** (1989), 195–218.
18. D.Mumford & J.Shah, *Optimal approximation by piecewise smooth functions and associated variational problems,* Comm. Pure Appl. Math., **XLII** (1989), 577–685.