Global regularity in ultradifferentiable classes

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A linear partial differential operator P defined on an open set Ω of $\mathbb{R}^N$ with coefficients in $C^\infty(\Omega)$ (resp., in $\mathcal{A}(\Omega)$) is said to be locally hypoelliptic (resp., locally analytic hypoelliptic) in Ω if for every $f \in C^\infty(U)$ (resp., $f \in \mathcal{A}(U)$), with $U \subset \Omega$ any open set, all the solutions $u \in \mathcal{D}'(U)$ of $Pu = f$ belong to $C^\infty(U)$ (resp., to $\mathcal{A}(U)$). If P is defined on the torus $\mathbb{T}^N$, then P is said to be globally hypoelliptic (resp., globally analytic hypoelliptic) in $\mathbb{T}^N$ if all the solutions $u \in \mathcal{E}'(\mathbb{T}^N)$ of $Pu = f$ belong to $C^\infty(\mathbb{T}^N)$ (resp., to $\mathcal{A}(\mathbb{T}^N)$). We observe that the local hypoellipticity (resp., local analytic hypoellipticity) implies the global hypoellipticity (resp., global analytic hypoellipticity). By the celebrated sum of squares theorem of Hormander [9] the finite type condition is sufficient for the local hypoellipticity of P. But this condition is not sufficient for the local analytic hypoellipticity of P as it was first observed by Baouendi and Goulaouic [4]. Other classes of locally hypoelliptic operators which fail to be locally analytic hypoelliptic have been found and there are important results on analytic regularity. All such operators also *fail* to be locally hypoelliptic in the setting of ultradifferentiable function spaces (see, [1, Propositions 4.1 and 4.2] for example). Cordaro and Himonas [5] proved that the finite type condition is *sufficient* for the global analytic hypoellipticity of some classes of operators in the form of a sum of squares of vector fields with real valued and real analytic coefficients. This result was further extended in Refs. [6–8,10]. In particular, Himonas and Petronilho [7,10] showed the global analytic hypoellipticity (and also the global Gevrey hypoellipticity) of certain operators of the type $P = P(t, D_t, D_x)$ defined on the torus $\mathbb{T}^{m+n}$ with real valued coefficients in $\mathcal{A}(\mathbb{T}^m)$ and globally hypoelliptic in $\mathbb{T}^{m+n}$.

Motivated by the recent work developed in [5–8,10] and in [2,3], in the paper [1] we investigate the global hypoellipticity of linear partial differential operators defined on the torus $\mathbb{T}^N$ in a bigger scale of spaces, namely, in the setting of ultradifferentiable classes. Actually, we prove the ω-regularity of solutions of operators of type $P = P(t, D_t, D_x)$ defined on the torus $\mathbb{T}^{m+n}$ with real valued coefficients in $\mathcal{E}_*(\mathbb{T}^m)$ and which are globally hypoelliptic in $\mathbb{T}^{m+n}$. Therefore, we extend the previous work for Gevrey classes of Himonas and Petronilho [7,10] (see, [1, Theorem 3.1]). As a consequence, we obtain some applications to sublaplacians that may satisfy the finite type condition or may be of infinite type at most points, see [1, §4]. We also characterize the global ω-hypoellipticity of linear partial differential operators with constant coefficients in $\mathbb{T}^N$ in terms of the symbol, see [1, Proposition 3.1].

REFERENCES

1. A.A. Albanese, D. Jornet, *Global regularity in ultradifferentiable classes*, Ann. Mat. Pura Appl. 193 (2014), 369-387.
2. A.A. Albanese, D. Jornet, A. Oliaro, *Quasianalytic wave front sets for solutions of linear partial differential operators*, Integr. Equ. Oper. Theory 66 (2010), 153–181.
3. A.A. Albanese, D. Jornet, A. Oliaro, *Wave front sets for ultradistribution solutions of linear pertial differential operators with coefficients in non-quasianalytic classes*, Math. Nachr. 285 (2012), 411-425.
4. M.S. Baouendi, C. Goulaouic, *Nonanalytic–hypoellipticity for some degenerate elliptic operators*, Bull. Amer. Math. Soc. 78 (1972), 483–486.
5. P.D. Cordaro, A.A. Himonas, *Global analytic hypoellipticity for a class of degenerate elliptic operators on the torus*, Math. Res. Lett. 1 (1994), 501–510.
6. P.D. Cordaro, A.A. Himonas, *Global analytic regularity for sums of squares of vector fields*, Trans. Amer. Math. Soc. 350 (1998), 4993–5001.
7. A.A. Himonas, G. Petronilho, *On Gevrey regularity of globally C^∞ hypoelliptic operators*, J. Diff. Eq. 207 (2004), 267–284.
8. A.A. Himonas, G. Petronilho, *On C^∞ and Gevrey regularity of sublaplacians*, Trans. Amer. Math. Soc. 358 (2006), 4809–4820.
9. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Mat. 119 (1967), 147–171.
10. G. Petronilho, *On Gevrey solvability and regularity*, Math. Nachr. 282 (2009), 470–481.