Characterizing Fréchet-Schwartz spaces via power bounded operators

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A continuous linear operator T acting in a Fréchet space X is called *power bounded* (resp. *mean ergodic*, resp. *uniformly mean ergodic*) if the sequence $\{T^n\}_{n=1}^{\infty}$ of iterates (resp. the sequence of the Cesàro means $\{\frac{1}{n}\sum_{k=1}^{n} T^k\}_{n=1}^{\infty}$) is equicontinuous (resp. convergent for the strong operator toplogy τ_s, resp. convergent for the uniform operator topology τ_b). J. von Neumann (1931) proved that unitary operators in Hilbert spaces are mean ergodic. F. Riesz (1938) showed that every power bounded operator in an L^p-space ($1 < p < \infty$) is mean ergodic. In 1939 E.R. Lorch extended this result to all reflexive Banach spaces. It quickly became evident that there was an intimate connection between geometric properties of the underlying Banach space X and mean ergodic operators on X. Concerning the converse, in 1997 E.Yu.Emel'yanov showed that every Banach lattice with the property that every power bounded operator on the space is mean ergodic is necessarily reflexive, [6]. A major breakthrough came in 2001 when V.P. Fonf, M. Lin and P. Wojtaszczyk, [7], established the following characterizations for a Banach space X *with a basis*:

(i) X is finite-dimensional if and only if every power bounded operator on X is uniformly mean ergodic.

(ii) X is reflexive if and only if every power bounded operator on X is mean ergodic.

This paper initiated an immediate interest for analogous questions in the setting of Fréchet spaces.

The result of Emel'yanov was extended in [5], where it was shown that a Fréchet lattice X is reflexive if and only if every power bounded operator on X is mean ergodic. An analogue of (i) is also presented in [5]. Namely, a discrete Fréchet lattice X is Montel (i.e., bounded sets are relatively compact) if and only if every power bounded operator lying in the centre $Z(X)$ of X is uniformly mean ergodic. Concerning further results along the lines of (i) and (ii) above, it is known that a Fréchet space X *with a basis* is Montel if and only if every power bounded operator on X is uniformly mean ergodic, [2, Theorem 1.3], and that X is reflexive if and only if every power bounded operator on X is mean ergodic, [2, Theorem 1.4]. For analogous results in the setting of locally convex spaces we refer to [3]. If the Fréchet space X is not assumed to have a basis, then X is Montel if and only if every power bounded, mean ergodic operator defined on any closed subspace of X is uniformly mean ergodic, [2, Theorem 5.4], and X is reflexive if and only if every power bounded operator defined on any closed subspace of X is mean ergodic, [2, Proposition 5.1].

In a conference in Trier (Germany) in 2008, where the above mentioned results were presented for the first time, Prof. A. Pelczyński suggested that there should be similar criteria available which characterize *Fréchet-Schwartz* spaces. In order to be able to distinguish between "Montel and Schwartz" it is necessary to find an appropriate (and stronger) notion of operator convergence than τ_b-convergence. The aim of the paper [1] is to present such a notion and to invoke it to address Pelczynski's suggestion.

Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of continuous linear operators on a Fréchet space X, whose topology is generated by a fundamental, increasing sequence of seminorms $\{q_n\}_{n=1}^{\infty}$. Then $\{S_k\}_{k=1}^{\infty}$ is called *rapidly convergent* if there exists a continuous linear operator S on X such that, for every $n \in \mathbb{N}$ there exists $m > n$ with

$$\lim_{k\to\infty} \sup\{q_n((S_k - S)x) \colon q_m(x) \le 1\} = 0,$$

in which case we write $S_k \overset{(R)}{\to} S$ for $k \to \infty$. Whenever $S_k \overset{(R)}{\to} S$ for $k \to \infty$, then also τ_b-$\lim_{k\to\infty} S_k = S$. However, since there exist Fréchet-Montel spaces which fail to be Schwartz, τ_b-convergence of a sequence of operators need *not* imply its rapid convergence: this follows from the fact that τ_s-convergence of a sequence of operators in a Montel space implies its τ_b-convergence and from the characterization presented in [1, Corollary 3.4]. Namely, a Fréchet space $X \neq \{0\}$ is Schwartz if and only if every sequence of operators in X which is τ_s-convergent is also rapidly convergent.

An adequate response to Pelczyński's suggestion, via the notion of rapid convergence, is presented in the final two sections of paper [1]. In [1, Section 4] we introduce the new notion of an operator being rapidly mean ergodic. A deep result of S.F. Bellenot, [4], stating that each Fréchet-Schwartz space is a closed subspace of a Fréchet-Schwartz space with an *unconditional basis*, plays a role in establishing the main result of this section; see [1, Theorem 4.6]. Namely, let X be a Fréchet space which is a closed subspace of a Fréchet space with an unconditional basis. Then X is Schwartz if and only if every closed subspace Y of X has the property that every power bounded operator on Y is rapidly mean ergodic. In the final Section 5 of paper [1] this result is refined (see [1, Theorem 5.6]) for the important class of Fréchet spaces $\lambda_p(A)$, $p \in [1,\infty) \cup \{0\}$, known as *Köthe echelon spaces*, all of which have an unconditional basis. Indeed, it is shown that $\lambda_p(A)$ is Schwartz if and only if every power bounded operator on $\lambda_p(A)$ is rapidly mean ergodic.

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