Uniform convergence and spectra of operators in a class of Fréchet spaces

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Given a Banach space X and a continuous linear operator T on X, there are various classical results which relate conditions on the spectrum $\sigma(T)$ of T with the operator norm convergence of certain sequences of operators generated by T. For instance, if $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$, with $\|\ \|_{op}$ denoting the operator norm, (even $\frac{T^n}{n}\to 0$ in the weak operator topology suffices), then necessarily $\sigma(T)\subseteq\overline{\mathbb{D}}$, where $\mathbb{D}:=\{z\in\mathbb{C}\colon |z|<1\}$, [3, p.709, Lemma 1]. The stronger condition $\lim_{n\to\infty}\|T^n\|_{op}=0$ is equivalent to the requirement that both $\sigma(T)\subseteq\mathbb{D}$ and $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$ hold, [5]. An alternate condition, namely that $\{T^n\}_{n=1}^\infty$ is a convergent sequence relative to the operator norm, is equivalent to the requirement that the three conditions $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$, the range $(I-T)^m(X)$ is closed in X for some $m\in\mathbb{N}$ and $\Gamma(T)\subseteq\{1\}$ are satisfied, [8]. Here $\Gamma(T):=\sigma(T)\cap\mathbb{T}$ with $\mathbb{T}:=\{z\in\mathbb{C}\colon |z|=1\}$ being the boundary of $\mathbb{D}$. Such results as above are often related to the uniform mean ergodicity of T, meaning that the sequence of averages $\left\{\frac{1}{n}\sum_{m=1}^n T^m\right\}$ of T is operator norm convergent. For instance, if $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$ and $1\in\rho(T):=\mathbb{C}\setminus\sigma(T)$, then T is uniformly mean ergodic, [6, p.90, Theorem 2.7]. Or, if $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$, then T is uniformly mean ergodic if and only if $(I-T)(X)$ is closed, [7].

The first aim of paper [1] is to extend results of the above kind to the class of all Fréchet spaces referred to as prequojections; this is achieved in [1, Section 3]. The extension to the class of all Fréchet spaces is not possible as it is shown in [1, Proposition 3.10] and in [2, Example 3.11], for instance. We point out that a classical result of Katznelson and Tzafriri stating, for any operator T satisfying $\sup_{n\in\mathbb{N}}\|T^n\|_{op}<\infty$, that $\lim_{n\to\infty}\|T^{n+1}-T^n\|_{op}=0$ if and only if $\Gamma(T)\subseteq\{1\}$, [4], is also extended to prequojection Fréchet spaces; see [1, Theorem 3.33].

The second aim of paper [1] is inspired by well known applications of the above mentioned Banach space results to determine the uniform mean ergodicity of operators T which satisfy $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$ and belong to certain *operator ideals*, such as the compact or weakly compact operators; see, for example, [3, Ch. VIII, §8], [6, Ch. 2, §2.2], where T can even be quasi–compact. An extension of such a mean ergodic result to the class of quasi–precompact operators acting in various locally convex Hausdorff spaces is presented in [9]. For prequojection Fréchet spaces, this result is further extended to the (genuinely) larger class of quasi–Montel operators; see [1, Proposition 4.10, Remark 4.11 and Theorem 4.13]. A mean ergodic theorem for Cesàro bounded, weakly compact operators (and also reflexive operators) in a certain class of locally convex spaces (which includes all Fréchet spaces) is also presented in [1]; see [1, Proposition 4.1 and Remark 4.2(ii)].

REFERENCES

1. A.A. Albanese, J. Bonet, W.J. Ricker, *Uniform convergence and spectra of operators in a class of Fréchet spaces*, Abstract and Applied Anaysis, Article Id 179027 (2014), 1-16.
2. A.A. Albanese, J. Bonet, W.J. Ricker, *Convergence of arithmetic means of operators in Fréchet spaces.* J. Math. Anal. Appl. **401** (2013), 160–173.
3. N. Dunford, J.T. Schwartz, *Linear Operators I: General Theory.* (2nd printing), Wiley–Interscience, New York, 1964.
4. Y. Katznelson, L. Tzafriri, *On power bounded operators.* J. Funct. Anal. **68** (1986), 313–328.
5. J.J. Koliha, *Power convergence and pseudoinverses of operators in Banach spaces.* J. Math. Anal. Appl. **48** (1974), 446–469.
6. U. Krengel, *Ergodic Theorems.* de Gruyter, Berlin, 1985.
7. M. Lin, *On the uniform ergodic theorem.* Proc. Amer. Math. Soc. **43** (1974), 337–340.
8. M. Mbekhta, J. Zemànek, *Sur le théorème ergodique uniforme et le spectre.* C. R. Acad. Sci. Paris **317** (1993), 1155–1158.

9. V.A. Pietsch, *Quasi-präkompakte Endomorphismen und ein Ergodensatz in lokalkonvexen Vektorräumen.* J. Reine Angew. Math. **207** (1961), 16–30.