Uniform mean ergodicity of C_0–semigroups in a class of Fréchet spaces

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Let $(T(t))_{t\geq 0}$ be a 1–parameter C_0–semigroup of continuous linear operators in a Banach space X. Ergodic theorems have a long tradition and are usually formulated via existence of the limits of the Cesàro averages $C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt$, $r > 0$, or of the Abel averages $\lambda R_\lambda x = \lambda \int_0^\infty e^{-\lambda t}T(t)x\,dt$, $\lambda > 0$, for each $x \in X$, when $r \to \infty$ and $\lambda \to 0^+$, respectively. In the former case one speaks of the mean ergodicity of $(T(t))_{t\geq 0}$ and in the latter case of its Abel mean ergodicity. Of course, the above convergence is relative to the *strong operators topology* τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators on X. The following fundamental result characterizing the mean ergodicity (resp. Abel mean ergodicity) of $(T(t))_{t\geq 0}$ for the *operator norm convergence* in $\mathcal{L}(X)$, in which case one speaks of uniform mean ergodicity (resp. uniform Abel mean ergodicity), is due to M. Lin; see [4, Theorem & Corollary 1], [5, Theorem 12].

Theorem 0.1. *Let X be a Banach space and $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a strongly continuous C_0–semigroup with $T(0) = I$ satisfying $\lim_{t\to\infty} \left\|\frac{T(t)}{t}\right\| = 0$. The following assertions are equivalent.*

(1) $\lim_{r\to\infty} C(r)$ *exists for the operator norm topology in $\mathcal{L}(X)$.*

(2) *The range* $\mathrm{Im}A$ *of the infinitesimal generator A of $(T(t))_{t\geq 0}$ is a closed subspace of X.*

(3) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N R_1^n$ *exists for the operator norm topology in $\mathcal{L}(X)$.*

(4) *There exists a projection $P \in \mathcal{L}(X)$ with* $\mathrm{Im}P = \{x \in X : T(t)x = x\ \forall t \geq 0\}$ *such that* $\lim_{\lambda\downarrow 0^+} \|\lambda R_\lambda - P\| = 0$.

(5) $\lim_{n\to\infty}(\lambda R_\lambda)^n$ *exists for the operator norm topology in $\mathcal{L}(X)$ for (some) all $\lambda > 0$.*

(6) *There exists $\lambda_0 > 0$ such that*

$$\sup_{0<\lambda\leq\lambda_0} \|R_\lambda y\| < \infty, \quad y \in \overline{\mathrm{Im}A}.$$

Much of modern analysis occurs in locally convex Hausdorff spaces (briefly, lcHs) which are *non–normable*. The notions of a C_0–semigroup $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$, for X a Banach space, being mean ergodic or Abel mean ergodic relative to τ_s are purely topological and so carry over immediately to the setting when X is a lcHs. The natural analogue of the operator norm topology in $\mathcal{L}(X)$ is the topology τ_b of uniform convergence on the bounded subsets of the lcHs X. Accordingly, the notions of $(T(t))_{t\geq 0}$ being uniformly mean ergodic (resp. uniformly Abel mean ergodic), i.e., relative to τ_b, are also defined. The aim of the paper [1] is to clarify the role of Theorem 0.1 in the setting of lcHs'. Some relevant comments in this respect are appropriate.

Leaving the Banach space setting brings with it various inherent (unpleasant) features. For istance, given any strongly continuous C_0–semigroup $(T(t))_{t\geq 0}$ in a Banach space X there always exists $\omega > 0$ such that the semigroup $(e^{-\omega t}T(t))_{t\geq 0}$ is uniformly bounded, i.e., $\sup_{t\geq 0} e^{-\omega t}\|T(t)\| < \infty$. Already in non–normable (lc-)Fréchet spaces X this is *not* be the case (cf. [3], [2]), i.e., $(e^{-\omega t}T(t))_{t\geq 0}$ may fail to be an equicontinuous subset of $\mathcal{L}(X)$ for every $\omega > 0$. So, the general theory of C_0–semigroups in Fréchet spaces is more involved than in Banach spaces. The infinitesimal generator A of $(T(t))_{t\geq 0}$ is always a closed linear operator (not necessarily everywhere defined). In the Banach space setting the resolvent set $\rho(A)$ of A is always non–empty and open; not necessarily so if X is a Fréchet space, [2, Example 3.5(vii)]. It can even happen, for X a non–normable Fréchet space, that $\rho(A) = \emptyset$; this is established in [1, Proposition 4.6]. Moreover, some of the basic techniques for Banach spaces which are crucial for establishing variuos uniform mean ergodic theorems (e.g., if the resolvent operators of A satisfy $\|R(\lambda, A)\| \to 0$ as $\lambda \to 0+$ then $\|R(\lambda, A)\| < 1$ for all λ small enough, or if $S \in \mathcal{L}(X)$ satisfies $\|I - S\| < 1$ that S is invertible in $\mathcal{L}(X)$, or the inequality $\mathrm{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda,A)\|}$ for $\lambda \in \rho(A)$, or that $\rho(A)$ is the natural (open) domain in which $R(\cdot, A)$ is holomorphic) are not always available in non–normable Fréchet spaces. So, one cannot expect Theorem 0.1 to carry over to general lcHs X. In fact, it does not even extend to general Fréchet spaces as [1, Example 3.7] shows.

Despite the negative comments made above it turns out, nevertheless, that Theorem 0.1 *does* have a natural extension to an important and non–trivial class of Fréchet spaces, namely the *quojections*. Precisely, in [1, Theorem 3.2] it is shown that

Theorem 0.2. *Let X be a quojection Fréchet space and $(T(t))_{t\geq 0}$ be a locally equicontinuous, C_0–semigroup on X satisfying τ_b-$\lim_{t\to\infty} \frac{T(t)}{t} = 0$. Then the following assertions are equivalent.*

(1) *The semigroup $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

(2) *The infinitesimal generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ has closed range.*

(3) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for every $\lambda > 0$.*

(4) *The operator $\lambda R(\lambda, A)$ is uniformly mean ergodic for some $\lambda > 0$.*

(5) *The semigroup $(T(t))_{t\geq 0}$ is uniformly Abel mean ergodic.*

(6) *The sequence of iterates $\{(\lambda R(\lambda, A))^n\}_{n=1}^{\infty}$ converges in $\mathcal{L}_b(X)$ for every (some) $\lambda > 0$.*

(7) $\overline{\mathrm{Im} A}$ *is a quojection and there exists $\lambda_0 > 0$ such that*

$$\{R(\lambda, A)y : \ y \in (0, \lambda_0]\} \in \mathcal{B}(X), \quad y \in \overline{\mathrm{Im} A}.$$

Recall that all Banach spaces, all countable products of Banach spaces, and many more Fréchet spaces are quojections. Concrete examples of quojections include the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$, the function spaces $L^p_{\mathrm{loc}}(\Omega)$, with $1 \leq p \leq \infty$ and $\Omega \subseteq \mathbb{R}^N$ and open set, and $C^{(m)}(\Omega)$ with $m \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^N$ an open set, when equipped with their canonical lc–topology. As alluded to above, Theorem 0.2 is the main result of the paper [1]. A further version of Theorem 0.2 is also presented in Section 3 of [1], namely to the class of *prequojection* Fréchet spaces (which properly contains the quojections). Section 2 of [1] is devoted to establishing variuos preliminary results needed in the sequel, many of interest in their own right. The final Section 4 of [1] presents some examples of concrete C_0–semigroups acting in particular quojection Fréchet spaces, with the aim of determining whether (or not) they are mean ergodic/uniformly mean ergodic.

REFERENCES

1. A.A. Albanese, J. Bonet, W.J. Ricker, *Uniform mean ergodicity of C_0–semigroups in a class of Fréchet spaces*, Funct. Approx. Comm. Math. 50 (2014), 307-349.
2. A.A. Albanese, J. Bonet, W.J. Ricker, *Montel resolvents and uniformly mean ergodic semigroups of linear operators*, Quaest. Math. 36 (2013), 253-290.
3. A.A. Albanese, J. Bonet, W.J. Ricker, *Mean ergodic semigroups of operators*, Rev. R. Acad. Cien. Serie A, Mat., RACSAM 106 (2012), 299-319.
4. M. Lin, *On the uniform ergodic theorem. II.* Proc. Amer. Math. Soc. 46 (1974), 217–225.
5. M. Lin, *Ergodic properties of an operator obtained from a continuous representation.* Ann. Inst. Henri Poincaré 13 (1977), 321–331.