Degenerate elliptic and parabolic operators on domains

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The study of degenerate elliptic operators started in the fifties and has been the object of many researches in a wide generality, since the seminal works of W. Feller in one dimension and J.J. Kohn and L. Nirenberg in higher dimensions. Of particular interest is the case of degeneracy at the boundary for second-order elliptic operators, and the results heavily depend on the behaviour of the coefficients at the boundary, i.e., on the order and the direction of degeneracy. A challenging borderline case occurs if the diffusion coefficients fully degenerate of first order in the normal direction to the boundary. In this case the *drift term* in normal direction is (roughly speaking) of the same order as the *diffusion part* and the sign and size of the drift coefficients play a crucial role. Here the model problem is given by the operator

$$L = -y\Delta + b_0 \cdot \nabla_x + b\partial_y \tag{1}$$

on the halfspace $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, y > 0\}$, where $b_0 \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$ and $\Delta = \Delta_x + \partial_y^2$. It has been shown in [2] that $-L$ with domain $D(L) = \{u \in W^{1,p}(\mathbb{R}^n_+) \,|\, u(\cdot, 0) = 0,\ Lu \in L^p(\mathbb{R}^n_+)\}$ generates an analytic semigroup on $L^p(\mathbb{R}^n_+)$ if $b > -1/p$. The techniques of [2] heavily depend on the condition $b > -1/p$, which allows to control gradient terms by the operator L through a Hardy type inequality which seems to fail for $b \le -1/p$ and without the Dirichlet boundary conditions embodied in the domain $D(L)$. In [3] we study the one dimensional case, i.e., the operator

$$L = -xD^2 + bD, \qquad \text{for all } b \in \mathbb{R}.$$

We refer to the 2013 Annual report for a description of these results. Starting from the 1-dimensional results, in [4] we study wellposedness and regularity of elliptic and parabolic partial differential equations in $L^2(\Omega)$, where Ω is either the halfspace or a bounded smooth domain, assuming that the second order coefficients degenerate at the boundary of first order. In particular, we establish that $-L$ generates an analytic C_0-semigroup on L^2 for each $b < -1/2$, endowed with the domain

$$D_2 = \left\{u \in W^{1,2}(\mathbb{R}^{N+1}_+) \cap W^{2,2}_{\rm loc}(\mathbb{R}^{N+1}_+) :\ \sqrt{y}\,|\nabla u|,\ y|D^2u| \in L^2(\mathbb{R}^{N+1}_+)\right\}$$

which possesses optimal regularity, but imposes no boundary condition because the drift points outward and is large enough. In addition, the operator $(-L, D_2)$ is accretive for $b \le -1$ and $b_0 = 0$, but it fails to be (quasi) accretive for $b \in (-1, -1/2)$ and $b_0 = 0$. This indicates that one cannot use form methods here. Observe that our results complement those of [2] for $p = 2$ where the opposite condition $b > -1/2$ was assumed. Based on the properties of the model operator, we also treat the problem on a bounded domain Ω in $\mathbb{R}^{N+1}$. We study an operator L in nondivergence form given by

$$L = -\varrho \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + \sum_{i=1}^{N+1} b_i D_i,$$

with continuous diffusion and drift coefficients on $\overline{\Omega}$, where the normal component of the drift is strictly less than $-1/2$ times the normal component of the matrix of the diffusion coefficients. We then show that $-L$ generates an analytic semigroup on $L^2(\Omega)$ when equipped with the domain

$$D_2^\Omega = \left\{u \in W^{2,2}_{\rm loc}(\Omega) \cap W^{1,2}(\Omega) : \varrho|D^2u| \in L^2(\Omega)\right\}$$

having optimal regularity and no boundary conditions, provided that

$$\kappa := \max_{\xi\in\partial\Omega} \frac{\langle b(\xi), \nu(\xi)\rangle}{\langle a(\xi)\nu(\xi), \nu(\xi)\rangle} < -\frac{1}{2}\,.$$

(Here, ϱ is a smooth extension of the distance function to the boundary.) By standard semigroup theory, this generation result allows to solve the corresponding inhomogeneous parabolic partial differential equation in optimal regularity.

In [1] we study again a class of elliptic operators L that degenerate at the boundary of a bounded open set $\Omega \subset \mathbb{R}^d$ with $\partial\Omega$ of class C^∞, but in a different perspective, i.e., we assume that L possesses a symmetrising invariant measure μ. Such operators are associated with diffusion processes in Ω which are invariant for time reversal. Indeed, the operator

$$Lu = \frac{1}{2}\mathrm{Tr}\,[\sigma\sigma^* D^2 u] + \langle b, Du\rangle, \quad u \in C^2(\Omega)\,, \tag{2}$$

where $b : \overline{\Omega} \to \mathbb{R}^d$ is of class C^1, $\sigma : \overline{\Omega} \to L(\mathbb{R}^d)$ is continuous on $\overline{\Omega}$, of class $C^1(\Omega)$ and such that, setting $a = \sigma\sigma^*$, $\det a(x) > 0\ \forall\, x \in \Omega$, is the Kolmogorov operator associated with the diffusion process described by the stochastic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \qquad X(0) = x \in \Omega, \tag{3}$$

where W is a Brownian motion. If there exists $\rho \in C^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $aD\rho + (g - 2b)\rho = 0$, where $g_j = \sum_{i=1}^d D_i a_{ij}$, $j = 1, ..., d$, then $L^*\rho = 0$, and $\mu(dx) = \rho(x)dx$ is an invariant measure for X. In this case, the process $Y(t) + X(1-t)$ is a solution of the same SDE, the operator L in (2) is a *gradient system*, it can be written in the form $Lu = \frac{1}{2\rho}\mathrm{div}\,\big(\rho a Du\big)$ and is symmetric in $L^2(\Omega,\mu)$. After showing that the corresponding elliptic equation $\lambda u - Lu = f$ has a unique variational solution for any $\lambda > 0$ and $f \in L^2(\Omega,\mu)$, under further hypotheses we obtain new results for the characterization of the domain of L.

Theorem 1. *Let $Lu = \frac{1}{2}\alpha\Delta u + \langle b, Du\rangle$ and, beside the previous hypotheses, assume the following:*

1. $\alpha,\ b \in C^\infty(\overline{\Omega}),\ 0 \le \alpha$, *and* $\alpha(x) = 0 \quad \iff \quad x \in \partial\Omega$.
2. *There exists $\rho \in C^\infty(\Omega)$ such that $\rho \in L^1(\Omega)$ and the equation $\alpha(x)D\log\rho(x) + D\alpha(x) = 2b(x)$ holds in Ω.*

Then, the domain of the variational operator L in $L^2(\Omega,\mu)$ can be characterised as follows:

$$D(L) = \{u \in W^{2,2}_{loc}(\Omega) :\ Du \in L^2(\Omega,\mu;\mathbb{R}^d),\ \alpha\Delta u \in L^2(\Omega,\mu)\}. \tag{4}$$

The main point in the above result is the condition $Du \in L^2(\Omega,\mu;\mathbb{R}^d)$, which follows from the estimate $\|Du\|_2 \le c(\lambda)\|f\|_2$ for the solution of $\lambda u - Lu = f$. This last estimate is stronger than the usual one $\|\alpha^{1/2}Du\|_2 \le c(\lambda)\|f\|_2$. In the particular case $b = \frac{k}{2}D\alpha$, $k \ge 1$, $\rho = \alpha^{k-1}$ and less regularity is required to get the above characterisation of the domain, that in this case reads

$$D(L) = \{u \in W^{2,2}_{loc}(\Omega) :\ Du \in L^2(\Omega,\mu;\mathbb{R}^d),\ \alpha\Delta u \in L^2(\Omega,\mu)\}. \tag{5}$$

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